Hierarchical Gradient Domain Vector Field Processing

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Outline

Hierarchical Gradient Domain Vector Field Processing

1. Why are we interested in **vector field processing**?
2. How do we perform vector field processing in the **gradient domain**?
3. How do we design a **hierarchy** for vector field processing?
4. How do we make the hierarchical solver **converge** quickly?
5. Conclusion

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Motivation (Vector Fields)

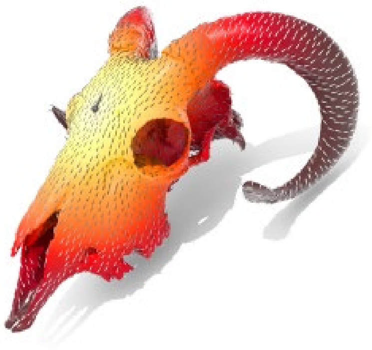
Vector Field Design

[M. Fisher, P. Schröder, M. Desbrun, H. Hoppe. SIGGRAPH 2007]

Operator Approach

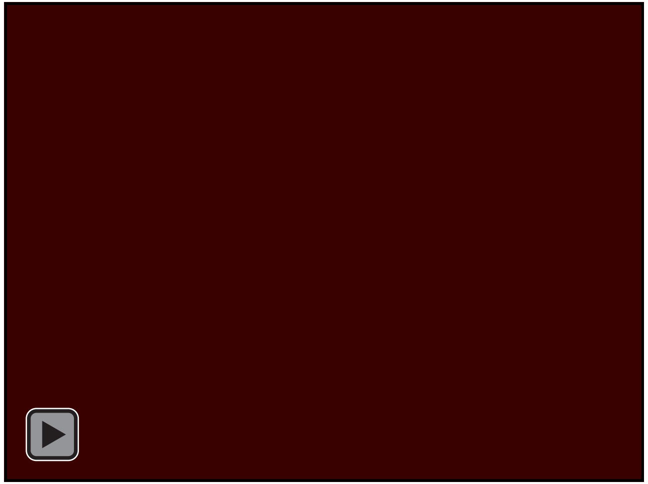
[O. Azencot, M. Ben-Chen, F. Chazal, M. Ovsjanikov. SGP 2013]

Vector Heat Method

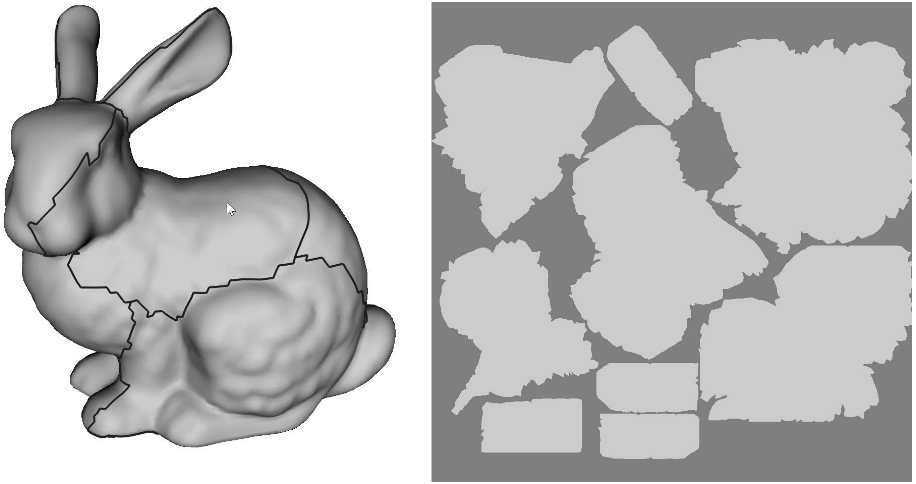


[N. Sharp, Y. Soliman, K. Crane. SIGGRAPH 2019]

Motivation (Scalar Fields)



[J. McCann, N. Pollard. SIGGRAPH 2008]



[F. Prada, M. Kazhdan, M. Chuang, H. Hoppe. SIGGRAPH 2018]

Hierarchy ⇒ Efficiency

Motivation (Hierarchical Approach)

Hierarchical approach + vector field processing

Carry over existing scalar field processing techniques

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Gradient Domain Formulation (0-Forms)

- In gradient-domain processing, we solve for a scalar field ϕ (a.k.a. 0-form) on a triangle mesh $\mathcal{M}$ by minimizing

$$E(\phi) = \boxed{\|\phi - \psi\|^2} + \alpha \boxed{\|d\phi - \nu\|^2}$$

where

- ψ: target field
- ν: target differential
- α: balancing weight

Gradient Domain Formulation (0-Forms)

$$E(\phi) = \|\phi - \psi\|^2 + \alpha\|d\phi - \nu\|^2$$

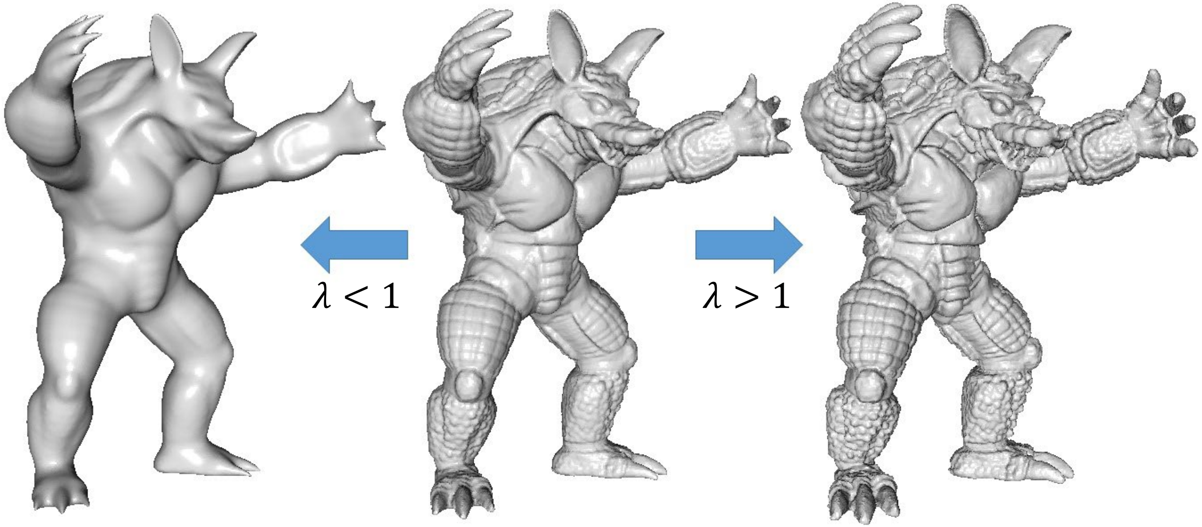
Describes traditional smoothing / sharpening:

$$\boxed{\nu = \lambda d\psi}$$

Gradient Domain Formulation (0-Forms)

$$E(\phi) = \|\phi - \psi\|^2 + \alpha\|d\phi - \nu\|^2$$

$$\boxed{\nu = \lambda d\psi}$$



[M. Chuang, S. Rusinkiewicz, M. Kazhdan. JCGT 2016]

Gradient Domain Formulation (0-Forms)

$$\arg\min_{\phi} E(\phi) = \|\phi - \psi\|^2 + \alpha\|d\phi - \nu\|^2$$

$\Downarrow$ Discretize using a basis

$$\mathbf{Ax} = \mathbf{b}$$

A Fourier Perspective

[P. Bhat, B. Curless, M. Cohen, C. L. Zitnick. ECCV 2008]

- In gradient-domain processing, we solve for a scalar field ϕ (a.k.a. 0-form) on a triangle mesh $\mathcal{M}$ by minimizing

$$E(\phi) = \|\phi - \psi\|^2 + \alpha\|d\phi - \nu\|^2$$

where

- ψ: target field
- ν: target differential
- α: balancing weight

If $\nu = d\eta$, η a 0-form

Then solving for ϕ is blending
- Low frequency components of ψ
- High frequency components of η

A Fourier Perspective

[P. Bhat, B. Curless, M. Cohen, C. L. Zitnick. ECCV 2008]

- In gradient-domain processing, we solve for a scalar field ϕ (a.k.a. 0-form) on a triangle mesh $\mathcal{M}$ by minimizing

$$E(\phi) = \|\phi - \psi\|^2 + \alpha\|d\phi - \nu\|^2$$

where

- ψ: target field (lower frequency)
- ν: target differential (higher frequency)
- α: balancing weight

Gradient Domain Formulation (1-Forms)

- In gradient-domain processing, we solve for a 1-form ω on a triangle mesh $\mathcal{M}$ by minimizing

$$E(\omega) = \boxed{\|\omega - \mu\|^2} + \alpha\boxed{\left(\|d\omega - \varrho\|^2 + \|\delta\omega - \varphi\|^2\right)}$$

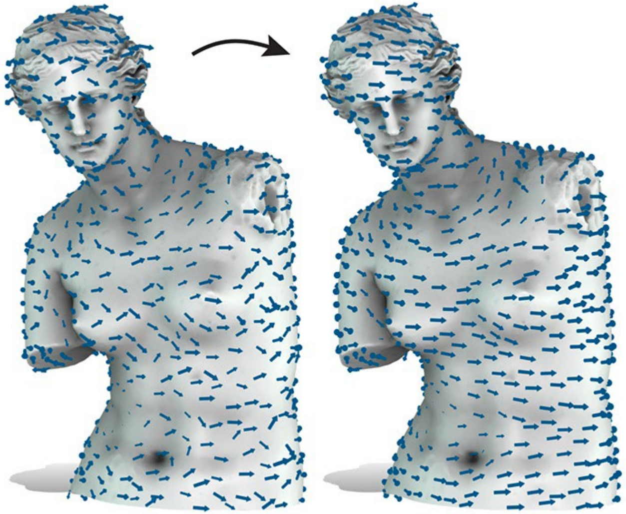
where

- μ: target field (lower frequency)
- ϱ: target differential (higher frequency – curl)
- φ: target co-differential (higher frequency – divergence)
- α: balancing weight

Gradient Domain Formulation (1-Forms)

$$E(\omega) = \|\omega - \mu\|^2 + \alpha(\|d\omega - \varrho\|^2 + \|\delta\omega - \varphi\|^2)$$

$$\boxed{\varrho = \lambda d\mu, \varphi = \lambda \delta\mu}$$



[O. Stein, M. Wardetzky, A. Jacobson, E. Grinspun. SGP 2020]

Gradient Domain Formulation (1-Forms)

$$\arg\min_{\omega} E(\omega) = \|\omega - \mu\|^2 + \alpha(\|d\omega - \varrho\|^2 + \|\delta\omega - \varphi\|^2)$$

$\Downarrow$ Discretize using a basis

$$\mathbf{Ax} = \mathbf{b}$$

$\mathbf{A}$ is Symmetric Positive Definite

Given the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, we are interested in solving it **efficiently**.

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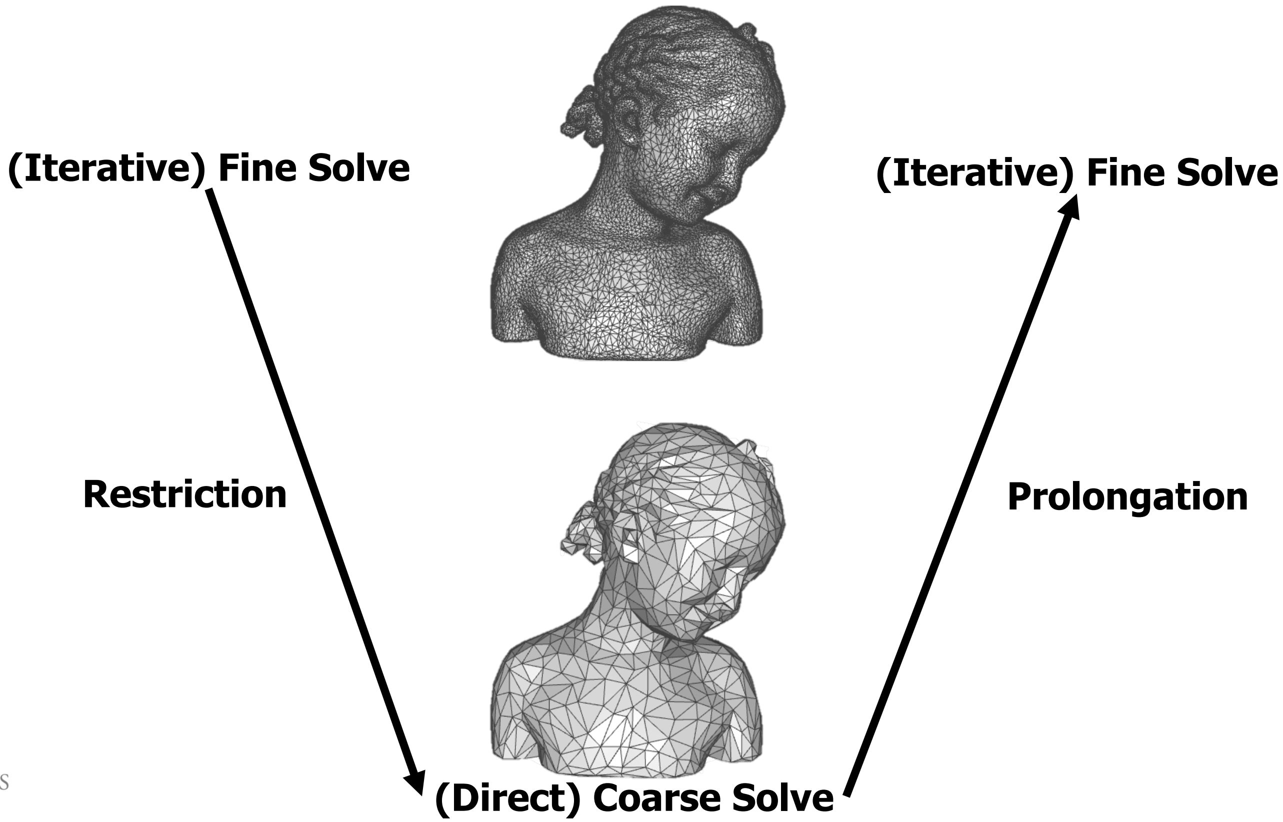
Hierarchical Approach

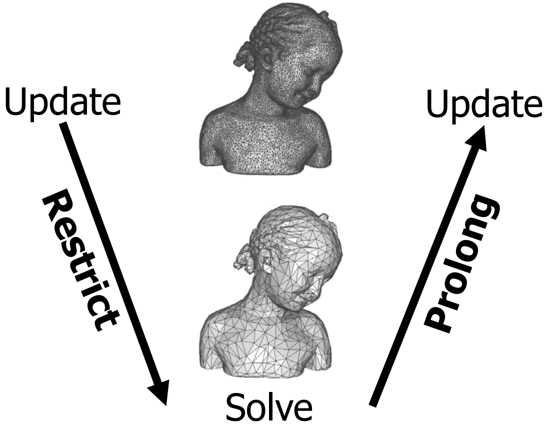


Prolongation Matrix

- $\mathbf{P} = \{P_{ij}\}$ Describes the coarse basis vectors as linear combination of the finer ones
- Fine basis: $\{\phi_i\}$, then coarse basis $\{\hat{\phi}_i\}$ with

$$\hat{\phi}_i = \sum_m P_{mi}\phi_m$$

- Roles:
  - Restrict from fine to coarse
  - Define coarse system matrix
  - Prolong from coarse to fine



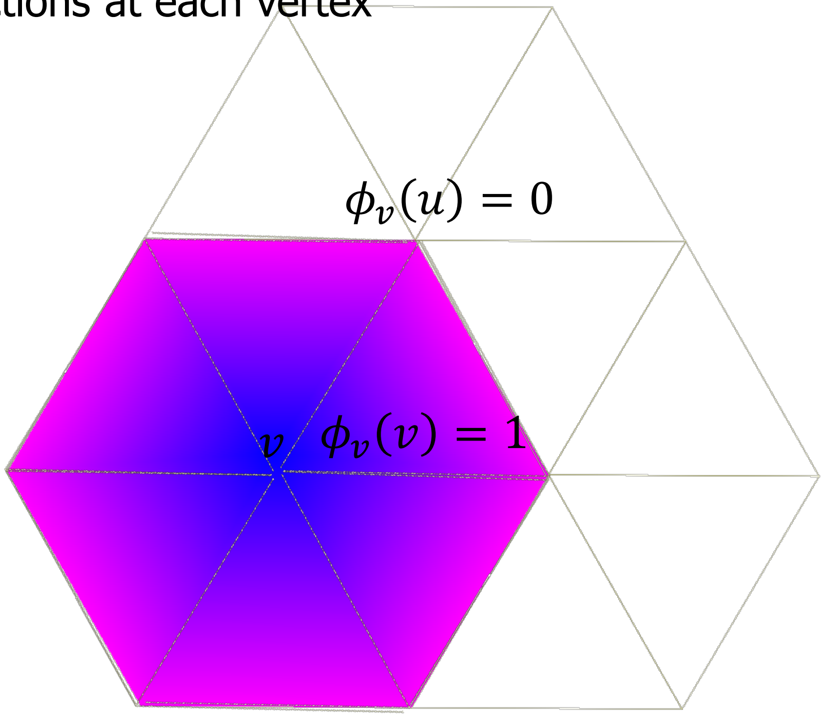
Which (fine) basis?

Which prolongation matrix?

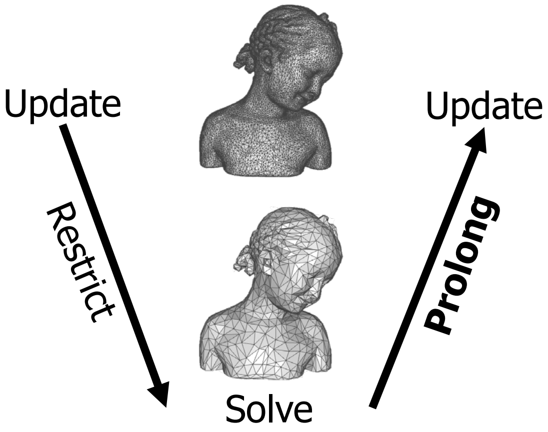
<u>Our approach:</u> 1-form basis that is constructed from a 0-form basis, and that induces a 1-form prolongation matrix.

0-Form Basis (Ω^0)

- $\{\phi_v\}$: hat basis functions at each vertex



0-Form Prolongation Matrix



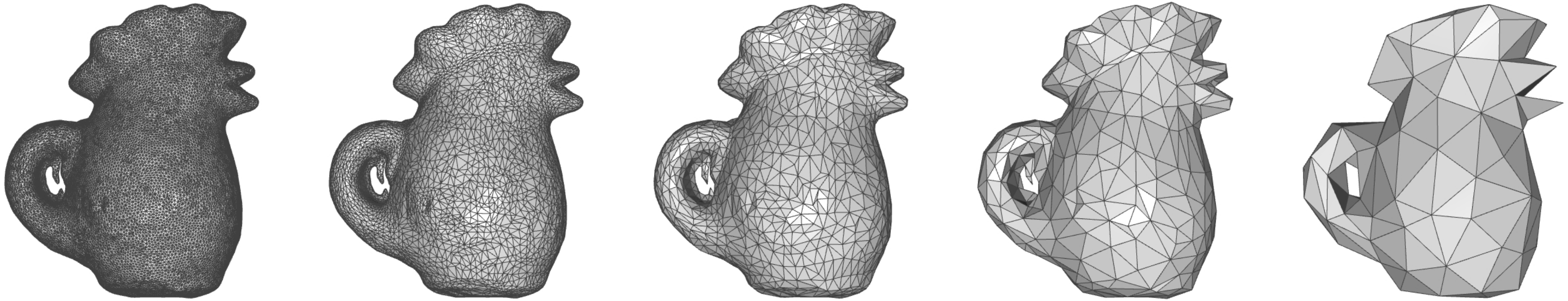
Given:

- Coarse and fine meshes
- A map from the fine to the coarse mesh

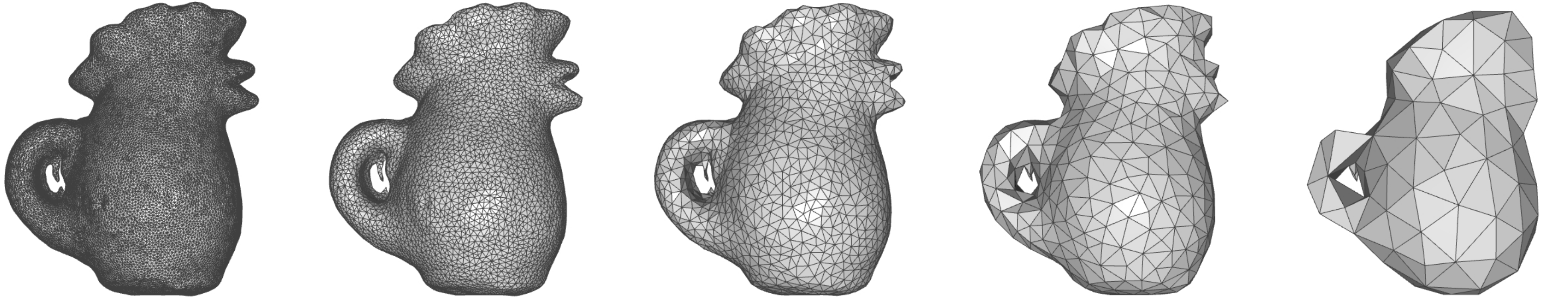
Goal:

- Pull back coarse basis functions to the fine mesh
- "Project" pulled back functions onto the fine basis

Existing Mesh Hierarchies



Mesh simplification [M. Garland, P. S. Heckbert. SIGGRAPH 1997]



Self-intrinsic parameterization [H. T. D. Liu, J. E. Zhang, M. Ben-Chen, A. Jacobson. SIGGRAPH 2021]

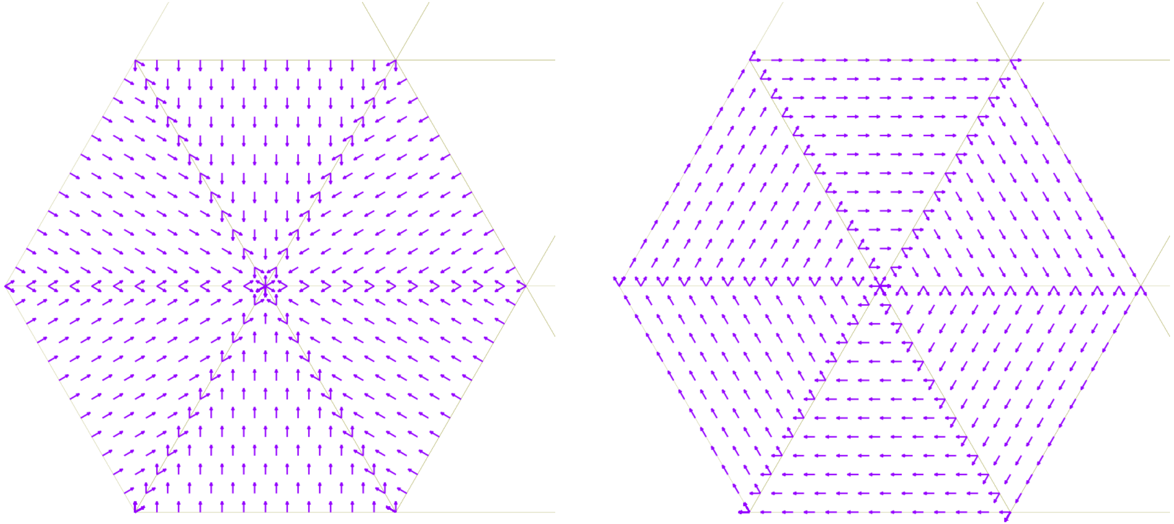
Which (fine) basis?

$$\Omega^0 \Rightarrow \Omega^1$$

Harmonic-Free 1-Form

- Given 0-form basis functions $\{\phi_i\}$, the harmonic-free basis $\{\omega_i\}$ is defined as

$$\omega_i = d\phi_i + \delta \star \phi_i$$

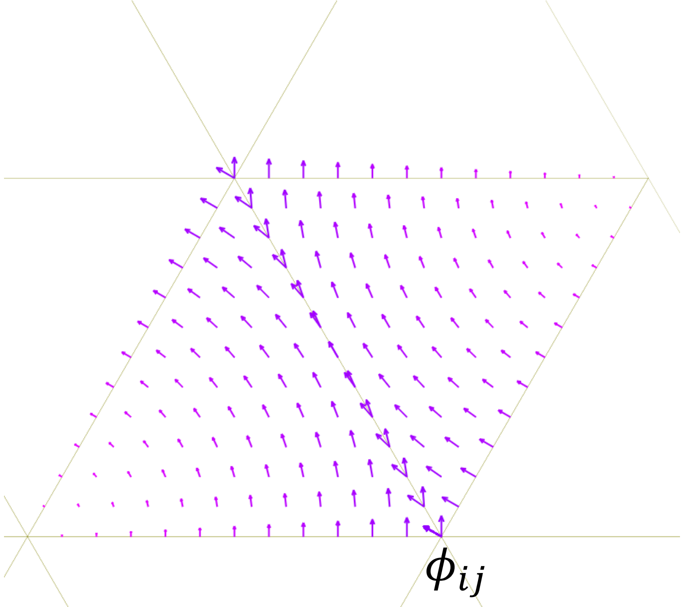


$d\phi_i$ $\qquad$ $\delta \star \phi_i$

Whitney 1-Form

- Given 0-form basis functions $\{\phi_i\}$, the Whitney 1-form basis $\{\phi_{ij}\}$ is defined as

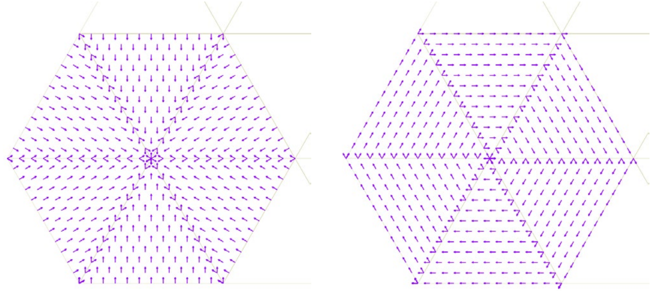
$$\phi_{ij} = \phi_i \cdot d\phi_j - \phi_j \cdot d\phi_i$$



Which prolongation matrix?

$$\mathbf{P}^0 \Rightarrow \mathbf{P}^1$$

Harmonic-Free 1-Form



$\omega_i = d\phi_i + \delta \star \phi_i$

<u>Given</u>:

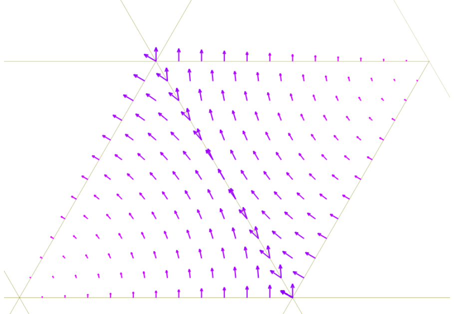
- Fine and coarse 0-form spaces: $\Omega^0, \widehat{\Omega}^0$ (defined by the 0-form prolongation $\mathbf{P}^0$)
- Harmonic-free 1-form space: $\Omega^1 \approx \Omega^0 \oplus \Omega^0$

<u>Goal:</u>

- We can also define a coarse 1-form space: $\widehat{\Omega}^1 \approx \widehat{\Omega}^0 \oplus \widehat{\Omega}^0$
- It induces a 1-form prolongation

$$\mathbf{P}^1 \approx \mathbf{P}^0 \oplus \mathbf{P}^0$$

Whitney 1-Form



$$\phi_{ij} = \phi_i \cdot d\phi_j - \phi_j \cdot d\phi_i$$

Given:

- Fine and coarse 0-form spaces: $\Omega^0, \widehat{\Omega}^0$ (defined by the 0-form prolongation $\mathbf{P}^0$)
- Whitney-free 1-form space: $\Omega^1 \approx \Omega^0 \wedge \Omega^0$

Goal:

- We can also define a coarse 1-form space: $\widehat{\Omega}^1 \approx \widehat{\Omega}^0 \wedge \widehat{\Omega}^0$
- It induces a 1-form prolongation

$$\mathbf{P}^1 \approx \mathbf{P}^0 \wedge \mathbf{P}^0$$

Generalization

Observation:

- 1-form space constructions
  - Hamonic-free 1-form: $\Omega_1 \approx \Omega_0 \bigoplus \Omega_0$
  - Whitney 1-form: $\Omega_1 \approx \Omega_0 \wedge \Omega_0$
- Induced 1-form prolongations:
  - Harmonic-free 1-form: $\mathbf{P}^1 \approx \mathbf{P}^0 \bigoplus \mathbf{P}^0$
  - Whitney 1-form: $\mathbf{P}^1 \approx \mathbf{P}^0 \wedge \mathbf{P}^0$

Generalization:

- $\mathcal{F}(V) = \bigoplus_i \bigotimes^{l_i} V, \mathcal{F}$ a functor on the category of vectors spaces
- $\mathcal{F}(\mathbf{P})$: an induced 1-form prolongation

Our Approach

Goal:

- Leverage existing 0-form hierarchies

Observation:

- Some existing 1-form spaces can be viewed as a multi-linear instances of a 0-form space

Show:

- Multi-linearity allows extending the 0-form prolongation to a 1-form prolongation

We propose using this to design a general vector field processing hierarchy that can reuse existing 0-form hierarchies

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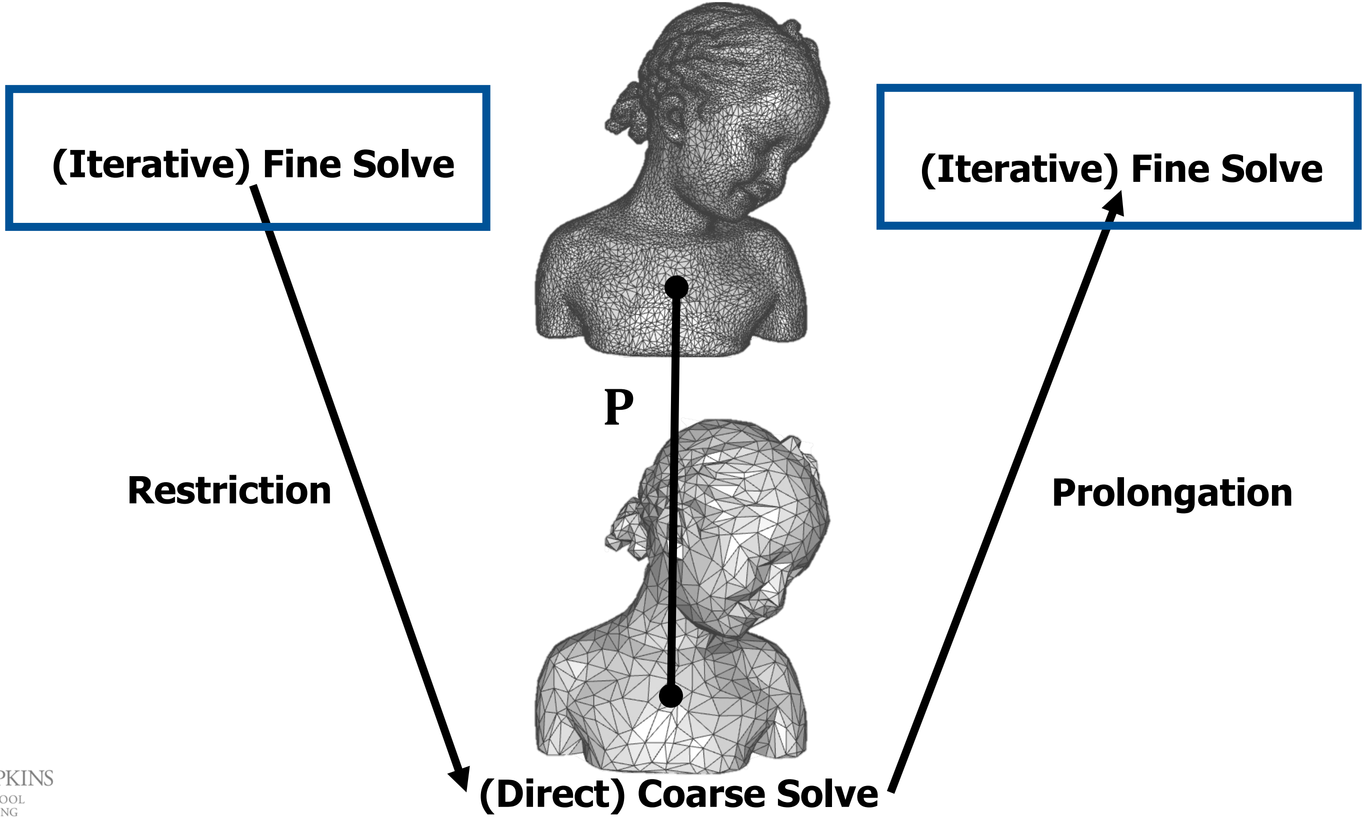
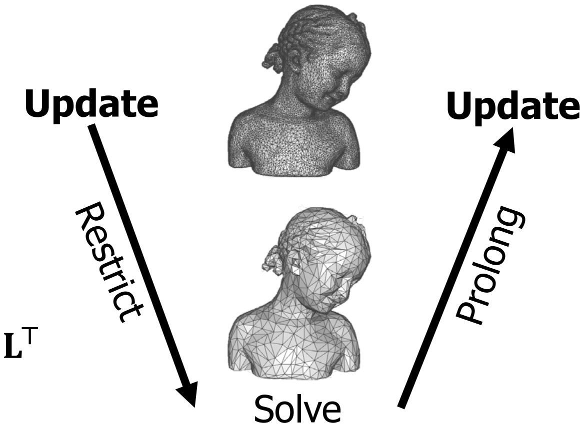
Hierarchical Approach



Relaxation Scheme



Split Methods:

Decompose $\mathbf{A}$ into diagonal and triangular components: $\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{L}^\top$

Iteratively update:

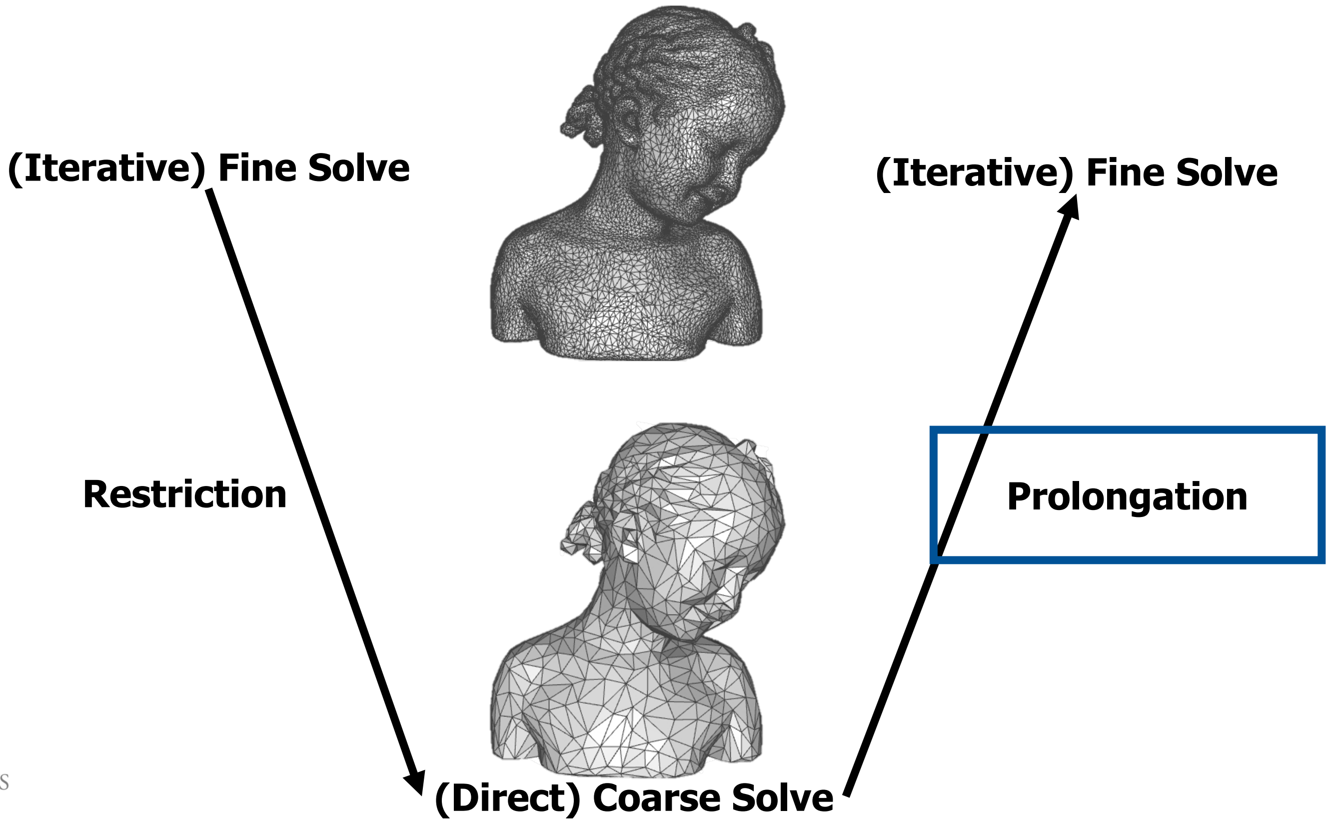
$$\mathbf{x} \leftarrow \mathbf{N}^{-1}(\mathbf{b} - (\mathbf{A} - \mathbf{N})\mathbf{x})$$

- Damped Jacobi $\quad \mathbf{N} = \frac{1}{\sigma}\mathbf{D}$
- Successive over-relaxation (SOR) $\quad \mathbf{N} = \mathbf{L} + \frac{1}{\sigma}\mathbf{D}$

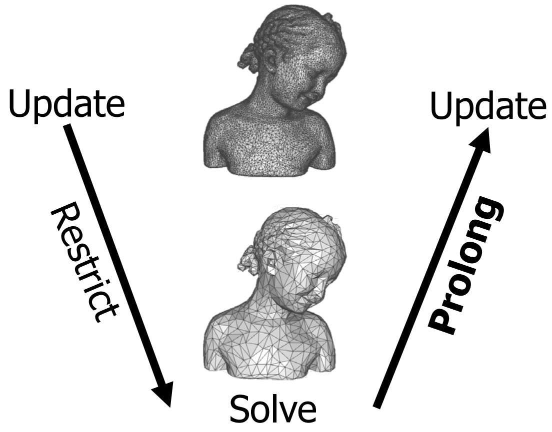
Exact Methods:

- Conjugate gradient

Hierarchical Approach



Smoothed Prolongation



- A typical step used in algebraic multigrid

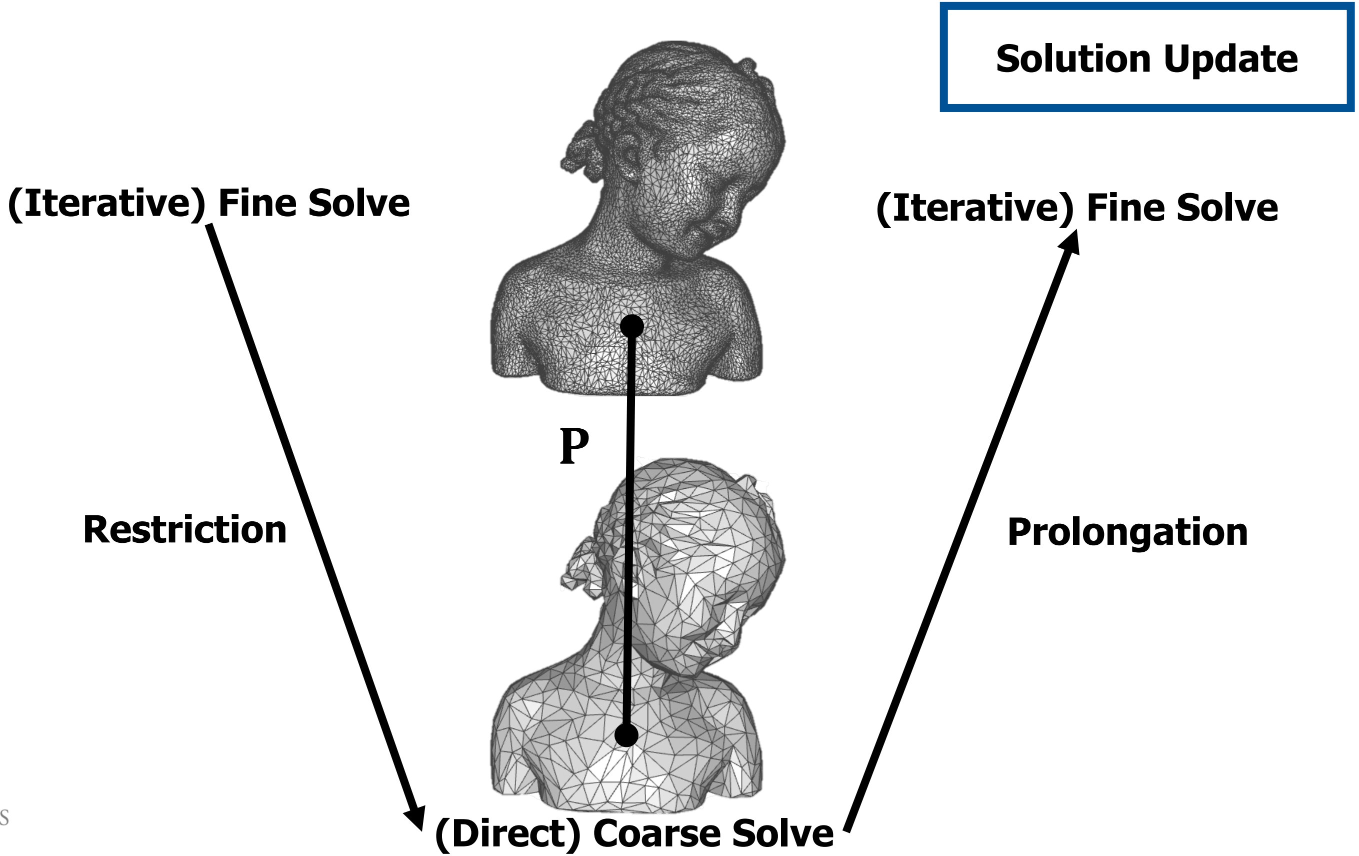
$$\mathbf{P} \leftarrow \mathbf{S}\mathbf{P}$$

$$\mathbf{S} = \mathbf{I} - \mathbf{N}^{-1}\mathbf{A}$$

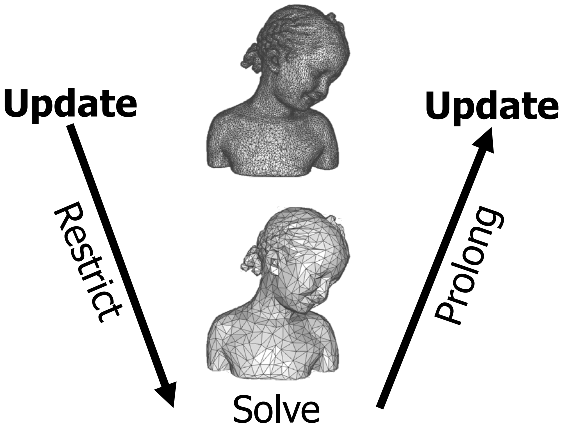
where $\mathbf{N}$ is defined as

- A relaxation smoothing step as in [P. Vanek, J. Mandel, M. Brezina. Computing 1996]
- To maintain the same sparsity, we discard matrix coefficients that are originally zeros

Hierarchical Approach



Solution Update



Inspiration:

- Krylov Subspace Method

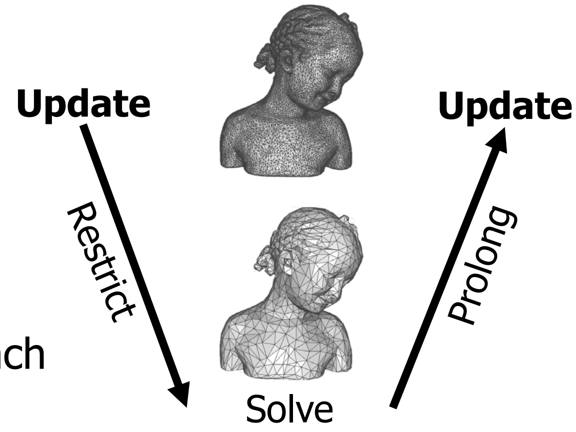
Given:

- $\mathbf{A}$: system matrix
- $\mathbf{b}$: right hand side

Goal:

- Construct the Krylov subspace $K_n \coloneqq \mathrm{Span}\{\mathbf{b}, \mathbf{Ab}, \mathbf{A}^2\mathbf{b}, \cdots \mathbf{A}^{n-1}\mathbf{b}\}$
- Find $\mathbf{x} \in K_n$ that best solves $\mathbf{Ax} = \mathbf{b}$

Krylov Subspace Update



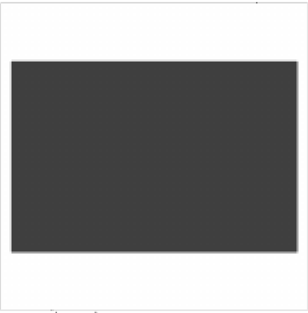
Idea:

- We construct the Krylov subspace using the estimate at each V-Cycle iteration:

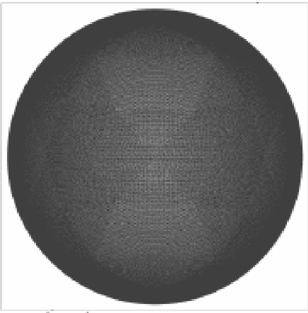
$$K_n = \mathrm{Span}\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n-1}\}$$

- Find $\mathbf{x} \in K_n$ that best solves $\mathbf{Ax} = \mathbf{b}$

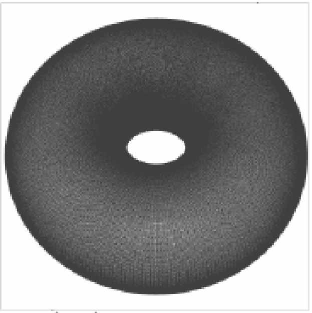
Models



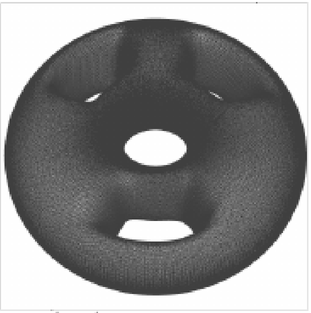
$|V| = 65,536$



$|V| = 122,882$



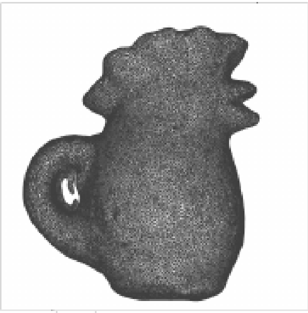
$|V| = 65,536$



$|V| = 65,530$

$|V| = 10,079$

$|V| = 74,764$



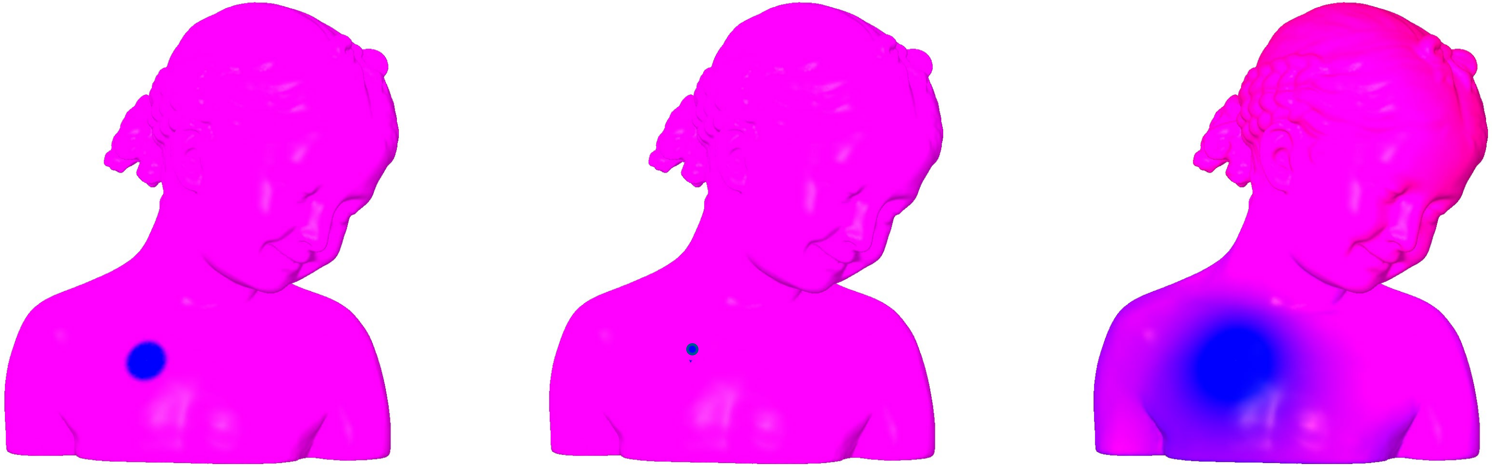
$|V| = 30,858$

$|V| = 59,994$

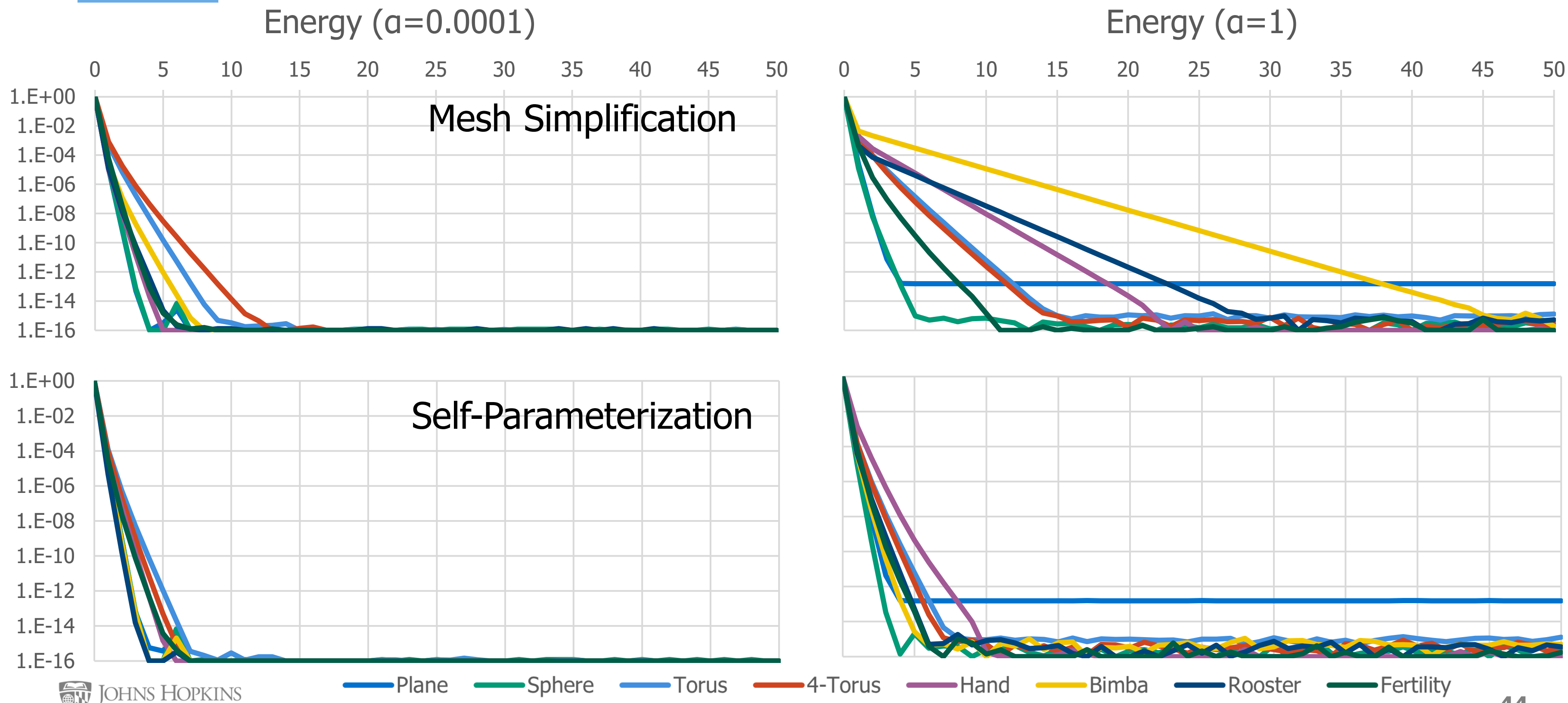
Convergence Study (0-Form System)

- An implicit scalar diffusion step ($\nu = 0$):

$$\arg\min_{\phi} E(\phi) = \|\phi - \psi\|^2 + \alpha\|d\phi\|^2$$



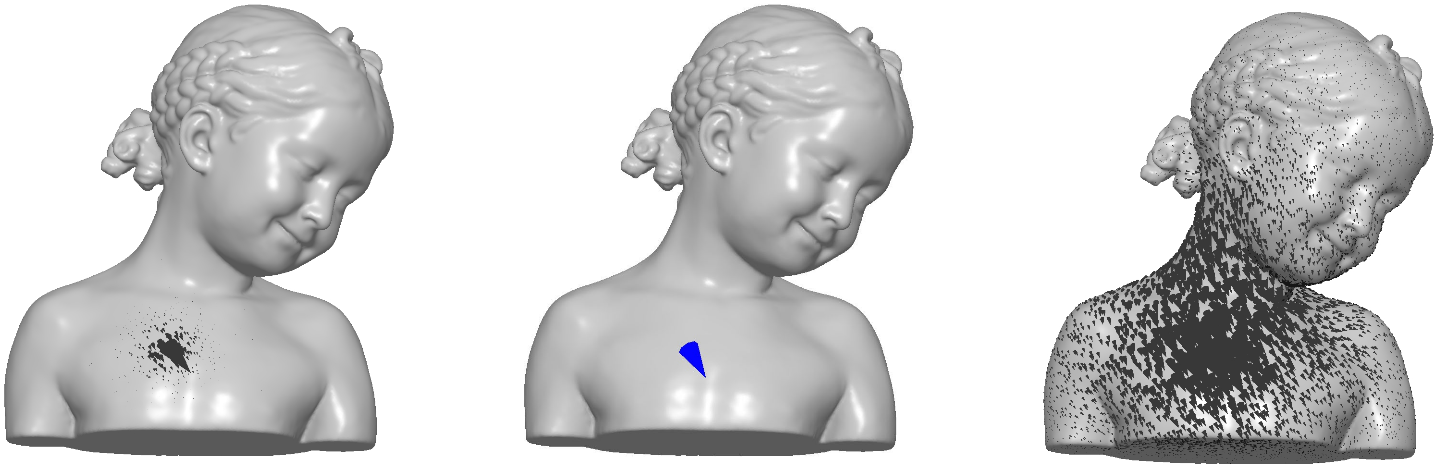
0-Form System Convergence



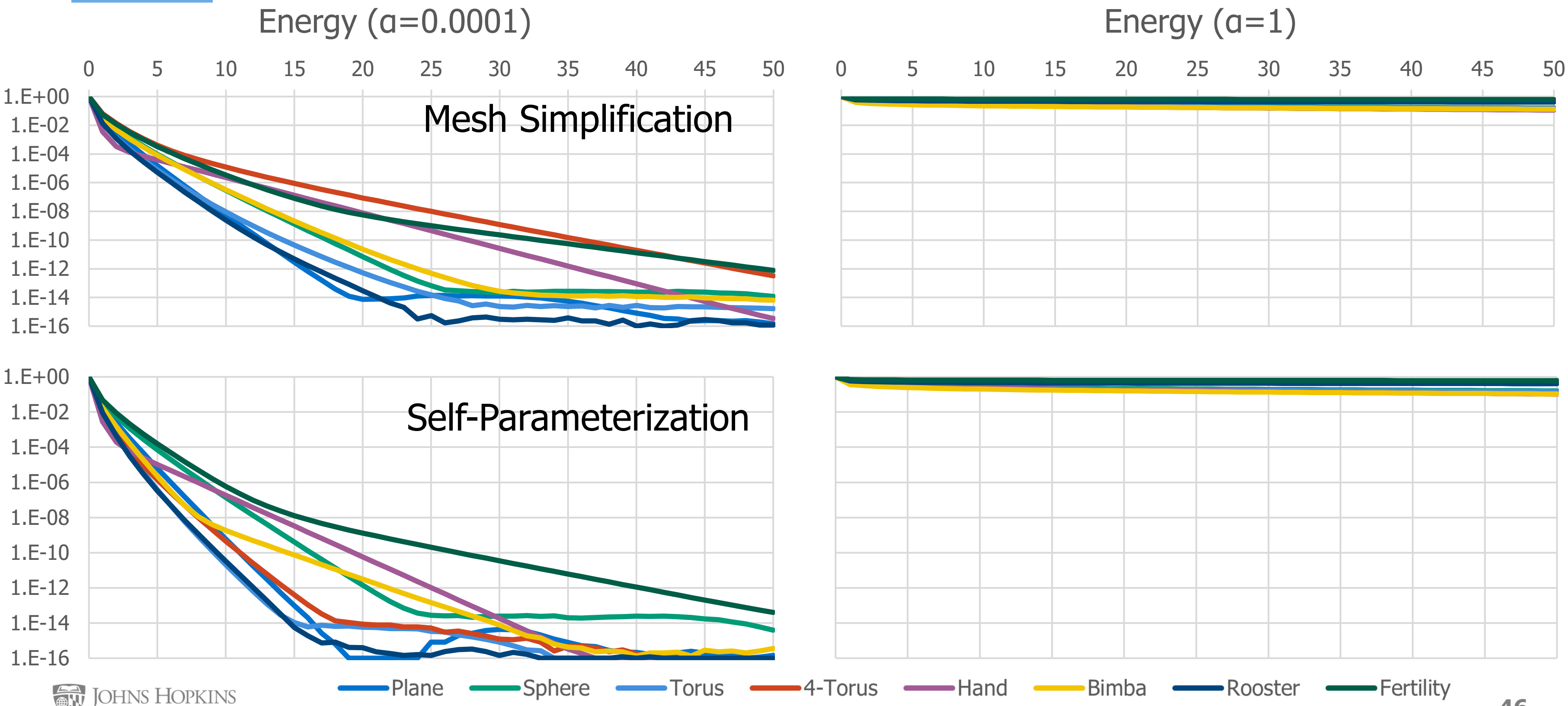
Convergence Study (1-Form System)

- An implicit vector diffusion step ($\varrho = 0$, $\varphi = 0$):

$$\arg\min_{\omega} E(\omega) = \|\omega - \mu\|^2 + \alpha(\|d\omega\|^2 + \|\delta\omega\|^2)$$



1-Form System Convergence



Convergence Comparison (0-Form/1-Form)

Iterations required to achieve errors $< 10^{-8}$ $(\alpha = 0.0001)$:

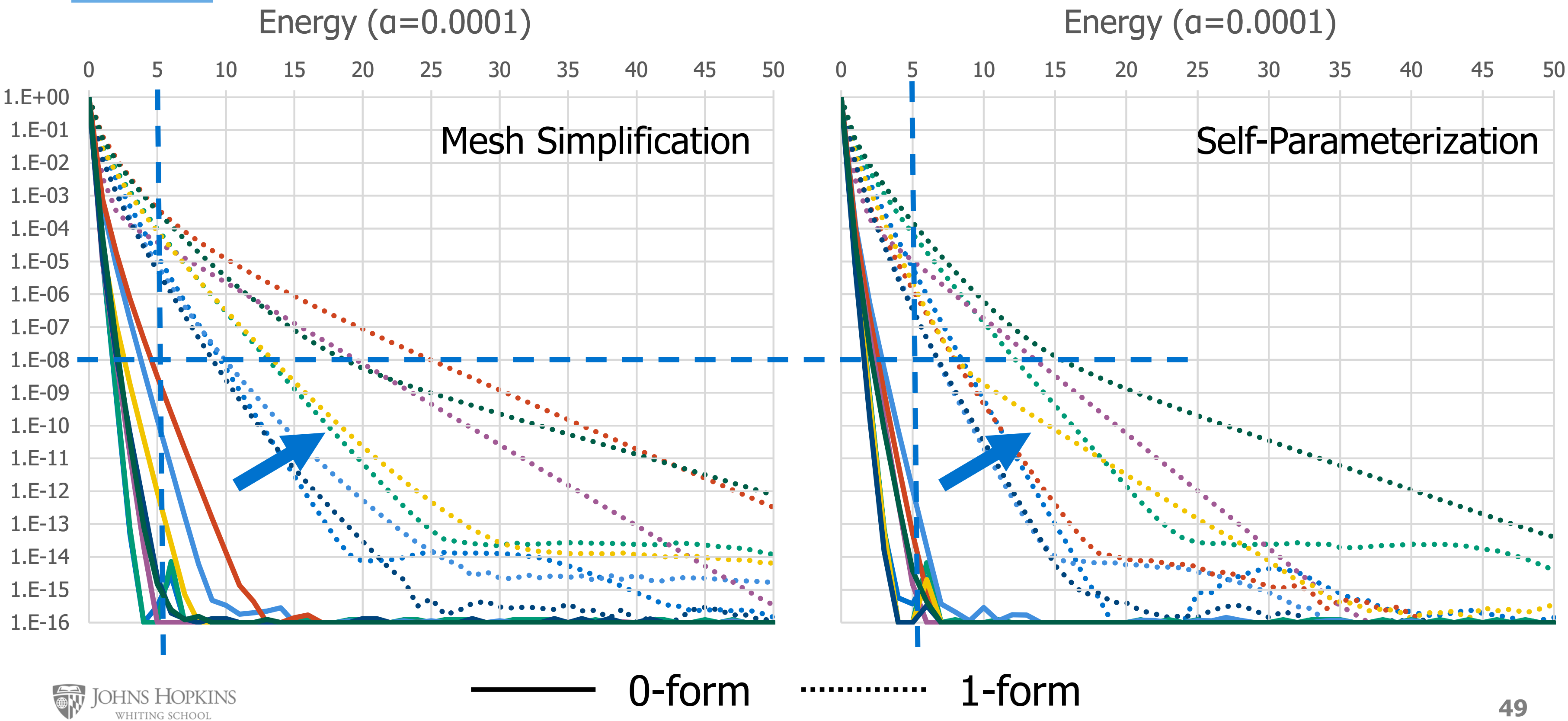
Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	Its $< 1 \cdot 10^{-8}$	Err at 50 its.	Its $< 1 \cdot 10^{-8}$
Plane	$1.00 \cdot 10^{-16}$ / $1.45 \cdot 10^{-16}$	2 / 9	$1.00 \cdot 10^{-16}$ / $1.43 \cdot 10^{-16}$	2 / 8
Sphere	$1.00 \cdot 10^{-16}$ / $1.18 \cdot 10^{-14}$	2 / 11	$1.00 \cdot 10^{-16}$ / $2.93 \cdot 10^{-15}$	2 / 11
Torus	$1.00 \cdot 10^{-16}$ / $1.66 \cdot 10^{-15}$	4 / 9	$1.00 \cdot 10^{-16}$ / $1.00 \cdot 10^{-16}$	3 / 6
4-Torus	$1.00 \cdot 10^{-16}$ / $3.22 \cdot 10^{-13}$	4 / 20	$1.00 \cdot 10^{-16}$ / $1.00 \cdot 10^{-16}$	4 / 7
Hand	$1.00 \cdot 10^{-16}$ / $3.42 \cdot 10^{-16}$	2 / 16	$1.00 \cdot 10^{-16}$ / $1.00 \cdot 10^{-16}$	2 / 11
Bimba	$1.00 \cdot 10^{-16}$ / $6.45 \cdot 10^{-15}$	3 / 12	$1.00 \cdot 10^{-16}$ / $3.54 \cdot 10^{-16}$	2 / 7
Rooster	$1.00 \cdot 10^{-16}$ / $1.21 \cdot 10^{-16}$	2 / 8	$1.00 \cdot 10^{-16}$ / $1.00 \cdot 10^{-16}$	2 / 6
Fertility	$1.00 \cdot 10^{-16}$ / $7.69 \cdot 10^{-13}$	2 / 15	$1.00 \cdot 10^{-16}$ / $4.02 \cdot 10^{-14}$	2 / 12

Convergence Comparison (0-Form/1-Form)

Iterations required to achieve errors $< 10^{-8}$ $(\alpha = 1)$:

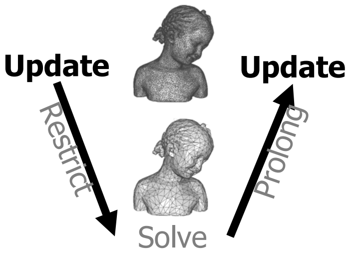
Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	Its $< 1 \cdot 10^{-8}$	Err at 50 its.	Its $< 1 \cdot 10^{-8}$
Plane	$1.00 \cdot 57^{-13}$ / $1.56 \cdot 10^{-1}$	2 / >50	$1.56 \cdot 10^{-13}$ / $1.43 \cdot 10^{-1}$	2 / >50
Sphere	$2.81 \cdot 10^{-16}$ / $1.64 \cdot 10^{-1}$	2 / >50	$4.64 \cdot 10^{-16}$ / $1.60 \cdot 10^{-1}$	2 / >50
Torus	$1.34 \cdot 10^{-15}$ / $1.58 \cdot 10^{-1}$	6 / >50	$1.30 \cdot 10^{-15}$ / $1.46 \cdot 10^{-1}$	3 / >50
4-Torus	$2.55 \cdot 10^{-16}$ / $4.47 \cdot 10^{-1}$	5 / >50	$3.15 \cdot 10^{-16}$ / $4.29 \cdot 10^{-1}$	3 / >50
Hand	$1.00 \cdot 10^{-16}$ / $1.15 \cdot 10^{-1}$	9 / >50	$1.00 \cdot 10^{-16}$ / $1.09 \cdot 10^{-1}$	4 / >50
Bimba	$1.96 \cdot 10^{-16}$ / $1.31 \cdot 10^{-1}$	18 / >50	$5.03 \cdot 10^{-16}$ / $1.07 \cdot 10^{-1}$	2 / >50
Rooster	$5.24 \cdot 10^{-16}$ / $4.18 \cdot 10^{-1}$	9 / >50	$3.67 \cdot 10^{-16}$ / $4.14 \cdot 10^{-1}$	3 / >50
Fertility	$1.00 \cdot 10^{-16}$ / $6.19 \cdot 10^{-1}$	3 / >50	$1.75 \cdot 10^{-16}$ / $6.15 \cdot 10^{-1}$	2 / >50

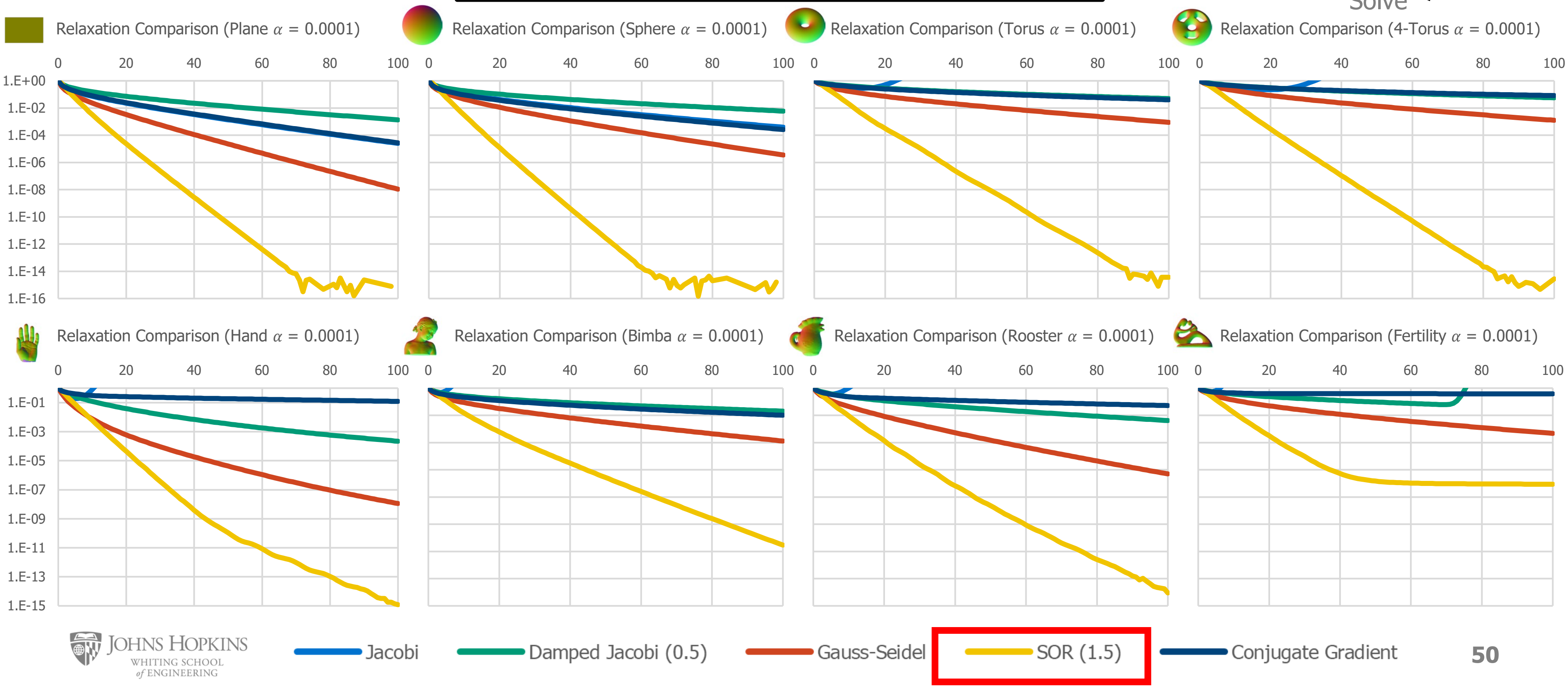
0-Form vs 1-Form



Relaxation Scheme Comparison

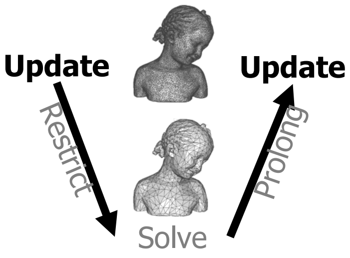
SOR ($\sigma = 1.5$) has the fastest convergence



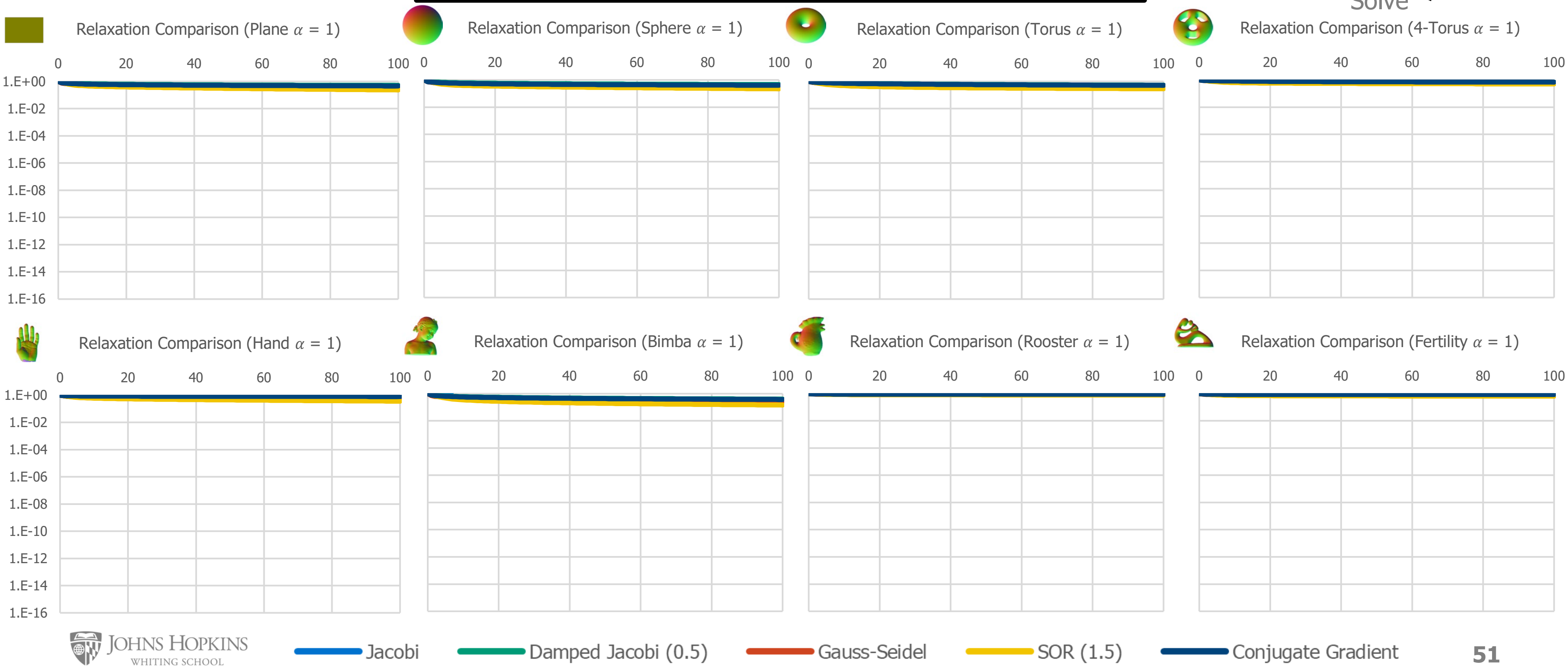


JOHNS HOPKINS
WHITING SCHOOL of ENGINEERING

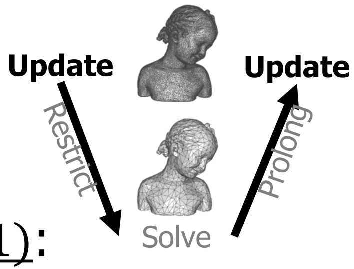
Relaxation Scheme Comparison



For big α, the relaxation scheme converge poorly



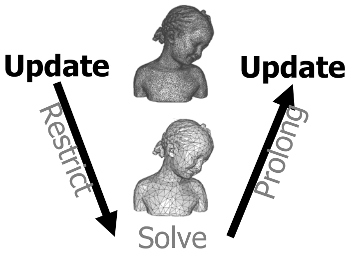
Fastest Relaxation Convergence



Iterations required to achieve errors $< 10^{-8}$ ($\alpha = 0.0001$):

Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	Its $< 1 \cdot 10^{-8}$	Err at 50 its.	Its $< 1 \cdot 10^{-8}$
Plane	$1.45 \cdot 10^{-16} \rightarrow 1.20 \cdot 10^{-16}$	9 → 6	$1.43 \cdot 10^{-16} \rightarrow 2.22 \cdot 10^{-16}$	8 → 5
Sphere	$1.18 \cdot 10^{-14} \rightarrow 1.00 \cdot 10^{-16}$	11 → 8	$2.93 \cdot 10^{-15} \rightarrow 1.00 \cdot 10^{-16}$	11 → 7
Torus	$1.66 \cdot 10^{-15} \rightarrow 1.00 \cdot 10^{-16}$	14 → 5	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	6 → 4
4-Torus	$3.22 \cdot 10^{-13} \rightarrow 1.00 \cdot 10^{-16}$	20 → 9	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	7 → 4
Hand	$3.42 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	16 → 6	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	11 → 5
Bimba	$6.45 \cdot 10^{-15} \rightarrow 1.65 \cdot 10^{-16}$	12 → 7	$3.54 \cdot 10^{-16} \rightarrow 1.67 \cdot 10^{-16}$	7 → 5
Rooster	$1.21 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	8 → 4	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	6 → 3
Fertility	$7.69 \cdot 10^{-13} \rightarrow 3.13 \cdot 10^{-16}$	15 → 8	$4.02 \cdot 10^{-14} \rightarrow 3.56 \cdot 10^{-16}$	12 → 8

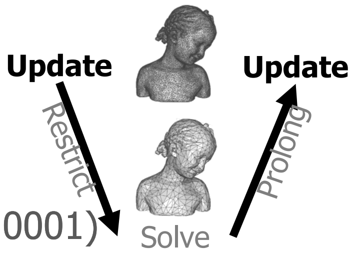
Fastest Relaxation Convergence

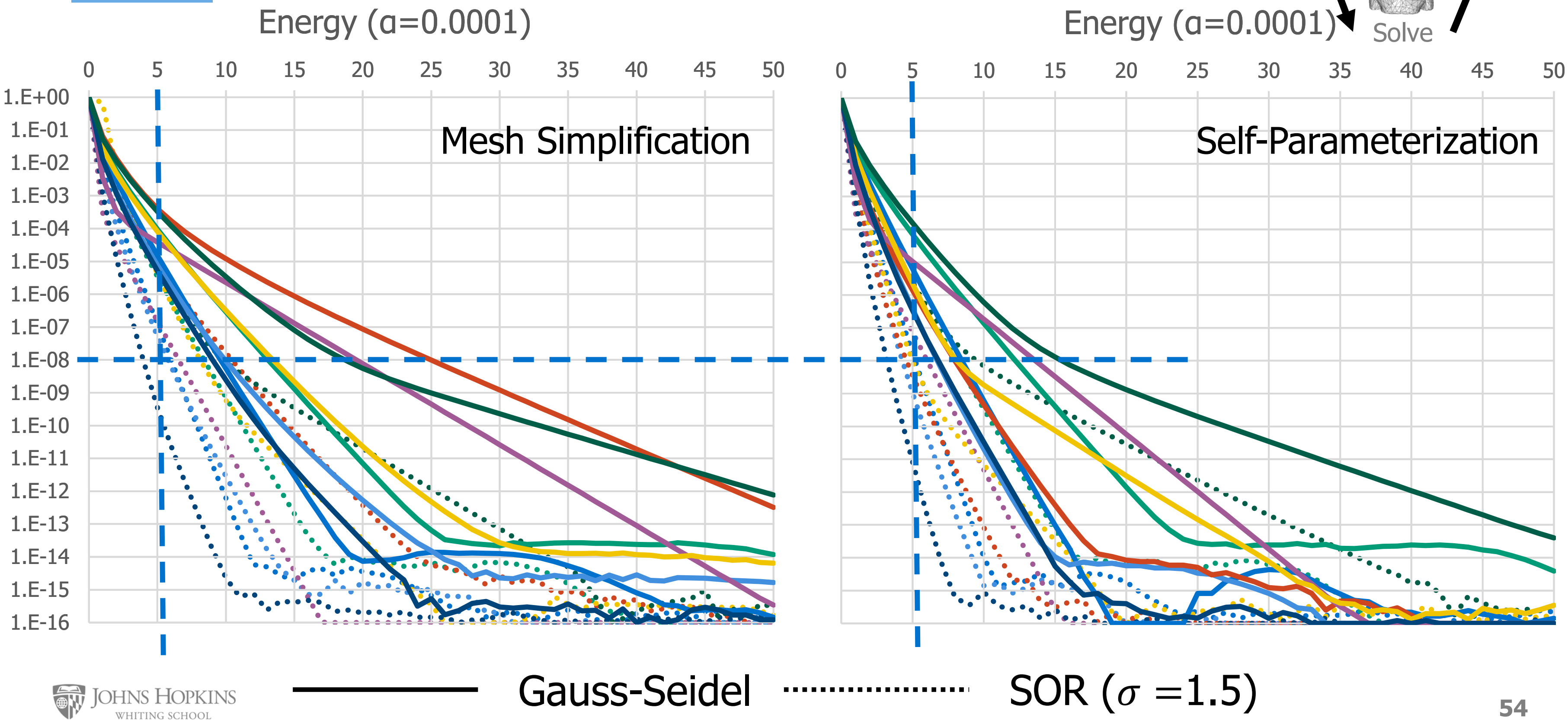


Iterations required to achieve errors $< 10^{-8}$ $(\alpha = 1)$:

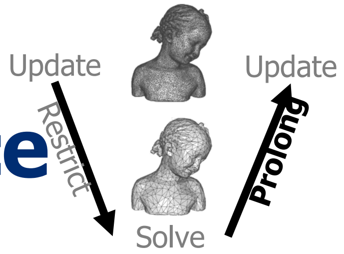
Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	**Its $< 1 \cdot 10^{-8}$**	**Err at 50 its.**	**Its $< 1 \cdot 10^{-8}$**
Plane	$1.56 \cdot 10^{-1} \rightarrow 1.34 \cdot 10^{-1}$	>50 → >50	$1.43 \cdot 10^{-1} \rightarrow 1.24 \cdot 10^{-1}$	>50 → >50
Sphere	$1.64 \cdot 10^{-1} \rightarrow 1.36 \cdot 10^{-1}$	>50 → >50	$1.60 \cdot 10^{-1} \rightarrow 1.33 \cdot 10^{-1}$	>50 → >50
Torus	$1.58 \cdot 10^{-1} \rightarrow 1.37 \cdot 10^{-1}$	>50 → >50	$1.46 \cdot 10^{-1} \rightarrow 1.29 \cdot 10^{-1}$	>50 → >50
4-Torus	$4.47 \cdot 10^{-1} \rightarrow 4.35 \cdot 10^{-1}$	>50 → >50	$4.29 \cdot 10^{-1} \rightarrow 4.17 \cdot 10^{-1}$	>50 → >50
Hand	$1.15 \cdot 10^{-1} \boxed{\rightarrow 7.29 \cdot 10^{-2}}$	>50 → >50	$1.09 \cdot 10^{-1} \boxed{\rightarrow 6.68 \cdot 10^{-2}}$	>50 → >50
Bimba	$1.31 \cdot 10^{-1} \rightarrow 1.03 \cdot 10^{-1}$	>50 → >50	$1.07 \cdot 10^{-1} \boxed{\rightarrow 8.78 \cdot 10^{-2}}$	>50 → >50
Rooster	$4.18 \cdot 10^{-1} \rightarrow 4.00 \cdot 10^{-1}$	>50 → >50	$4.14 \cdot 10^{-1} \rightarrow 3.98 \cdot 10^{-1}$	>50 → >50
Fertility	$6.19 \cdot 10^{-1} \rightarrow 6.06 \cdot 10^{-1}$	>50 → >50	$6.15 \cdot 10^{-1} \rightarrow 6.03 \cdot 10^{-1}$	>50 → >50

Fastest Relaxation Convergence





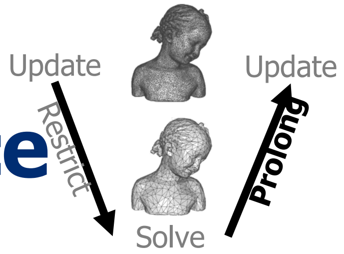
Smoothed Prolongation Convergence



Iterations required to achieve errors $< 10^{-8}$ ($\alpha = 0.0001$):

Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	Its $< 1 \cdot 10^{-8}$	Err at 50 its.	Its $< 1 \cdot 10^{-8}$
Plane	$1.45 \cdot 10^{-16} \rightarrow 1.90 \cdot 10^{-16}$	9 → 7	$1.43 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	8 → 6
Sphere	$1.18 \cdot 10^{-14} \rightarrow 1.00 \cdot 10^{-16}$	11 → 7	$2.93 \cdot 10^{-15} \rightarrow 1.00 \cdot 10^{-16}$	11 → 7
Torus	$1.66 \cdot 10^{-15} \rightarrow 1.14 \cdot 10^{-16}$	14 → 7	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	6 → 5
4-Torus	$3.22 \cdot 10^{-13} \rightarrow 2.92 \cdot 10^{-16}$	20 → 15	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	7 → 6
Hand	$3.42 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	16 → 15	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	11 → 7
Bimba	$6.45 \cdot 10^{-15} \rightarrow 5.4 \cdot 10^{-16}$	12 → 9	$3.54 \cdot 10^{-16} \rightarrow 9.18 \cdot 10^{-16}$	7 → 5
Rooster	$1.21 \cdot 10^{-16} \rightarrow 1.46 \cdot 10^{-16}$	8 → 7	$1.00 \cdot 10^{-16} \rightarrow 1.21 \cdot 10^{-16}$	6 → 5
Fertility	$7.69 \cdot 10^{-13} \rightarrow 3.61 \cdot 10^{-11}$	15 → 15	$4.02 \cdot 10^{-14} \rightarrow 1.99 \cdot 10^{-13}$	12 → 14

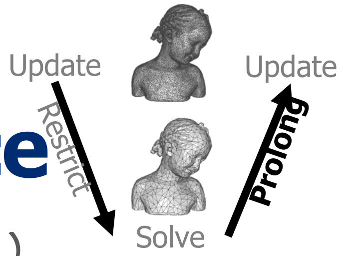
Smoothed Prolongation Convergence

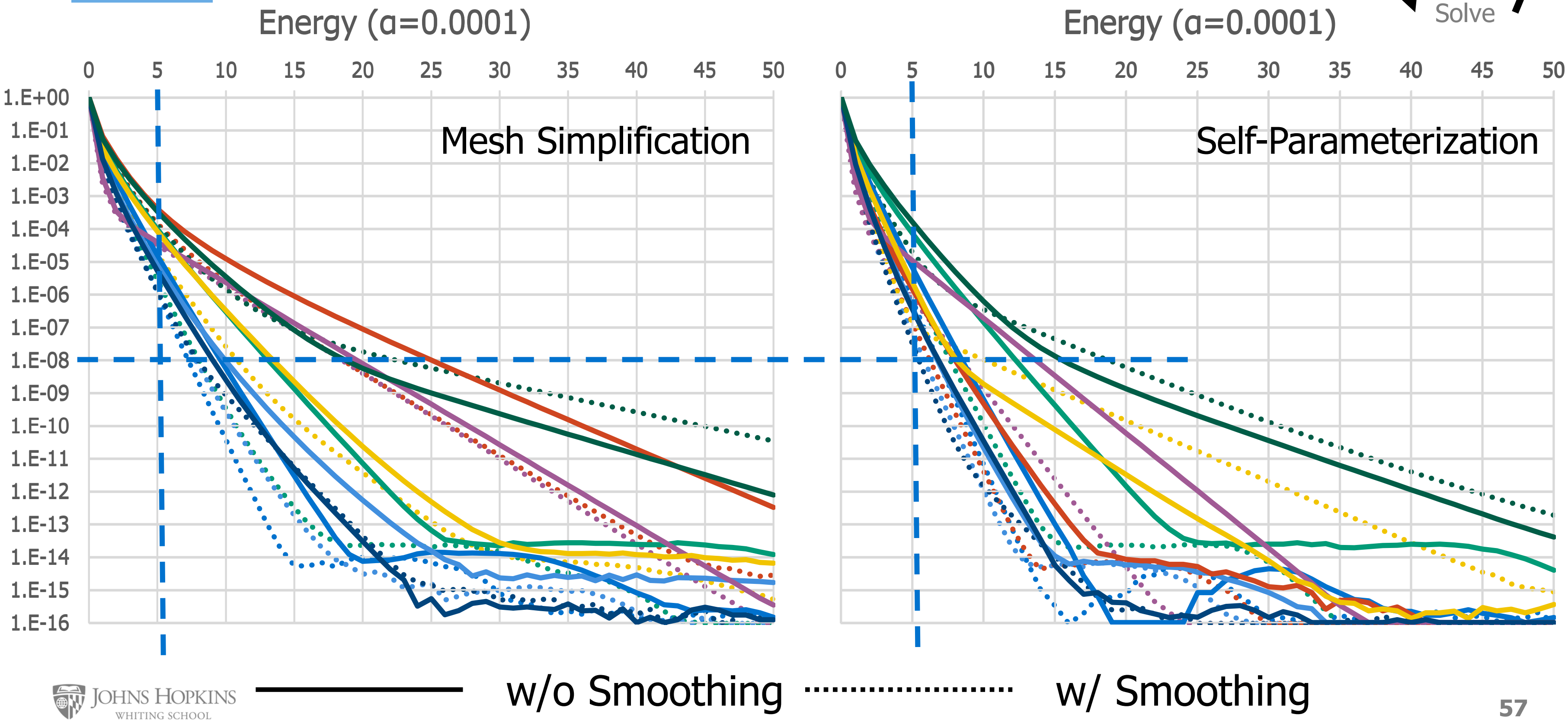


Iterations required to achieve errors $< 10^{-8}$ $(\alpha = 1)$:

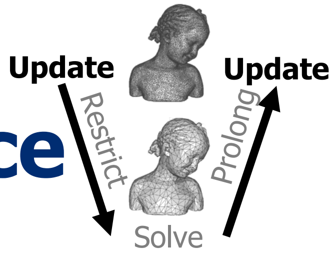
Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	**Its $< 1 \cdot 10^{-8}$**	**Err at 50 its.**	**Its $< 1 \cdot 10^{-8}$**
Plane	$1.56 \cdot 10^{-1} \rightarrow 1.35 \cdot 10^{-1}$	>50 ➔ >50	$1.43 \cdot 10^{-1} \rightarrow 1.25 \cdot 10^{-1}$	>50 ➔ >50
Sphere	$1.64 \cdot 10^{-1} \rightarrow 1.24 \cdot 10^{-1}$	>50 ➔ >50	$1.60 \cdot 10^{-1} \rightarrow 1.19 \cdot 10^{-1}$	>50 ➔ >50
Torus	$1.58 \cdot 10^{-1} \rightarrow 1.39 \cdot 10^{-1}$	>50 ➔ >50	$1.46 \cdot 10^{-1} \rightarrow 1.21 \cdot 10^{-1}$	>50 ➔ >50
4-Torus	$4.47 \cdot 10^{-1} \rightarrow 4.41 \cdot 10^{-1}$	>50 ➔ >50	$4.29 \cdot 10^{-1} \rightarrow 4.18 \cdot 10^{-1}$	>50 ➔ >50
Hand	$1.15 \cdot 10^{-1} \rightarrow 8.56 \cdot 10^{-2}$	>50 ➔ >50	$1.09 \cdot 10^{-1} \rightarrow 7.08 \cdot 10^{-2}$	>50 ➔ >50
Bimba	$1.31 \cdot 10^{-1} \rightarrow 1.09 \cdot 10^{-1}$	>50 ➔ >50	$1.07 \cdot 10^{-1} \rightarrow 7.84 \cdot 10^{-2}$	>50 ➔ >50
Rooster	$4.18 \cdot 10^{-1} \rightarrow 4.13 \cdot 10^{-1}$	>50 ➔ >50	$4.14 \cdot 10^{-1} \rightarrow 3.98 \cdot 10^{-1}$	>50 ➔ >50
Fertility	$6.19 \cdot 10^{-1} \rightarrow 6.13 \cdot 10^{-1}$	>50 ➔ >50	$6.15 \cdot 10^{-1} \rightarrow 6.05 \cdot 10^{-1}$	>50 ➔ >50

Smoothed Prolongation Convergence





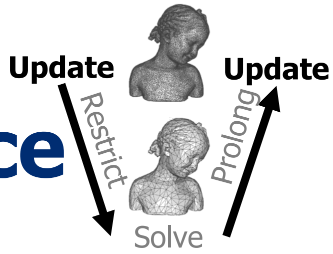
Krylov Subspace Update Convergence



Iterations required to achieve errors $< 10^{-8}$ ($\alpha = 0.0001$):

Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	Its $< 1 \cdot 10^{-8}$	Err at 50 its.	Its $< 1 \cdot 10^{-8}$
Plane	$1.45 \cdot 10^{-16} \rightarrow 1.20 \cdot 10^{-16}$	9 → 5	$1.43 \cdot 10^{-16} \rightarrow 2.22 \cdot 10^{-16}$	8 → 5
Sphere	$1.18 \cdot 10^{-14} \rightarrow 1.00 \cdot 10^{-16}$	11 → 6	$2.93 \cdot 10^{-15} \rightarrow 1.00 \cdot 10^{-16}$	11 → 6
Torus	$1.66 \cdot 10^{-15} \rightarrow 1.00 \cdot 10^{-16}$	14 → 5	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	6 → 4
4-Torus	$3.22 \cdot 10^{-13} \rightarrow 1.00 \cdot 10^{-16}$	20 → 8	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	7 → 5
Hand	$3.42 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	16 → 10	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	11 → 5
Bimba	$6.45 \cdot 10^{-15} \rightarrow 1.65 \cdot 10^{-16}$	12 → 6	$3.54 \cdot 10^{-16} \rightarrow 1.67 \cdot 10^{-16}$	7 → 4
Rooster	$1.21 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	8 → 5	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	6 → 4
Fertility	$7.69 \cdot 10^{-13} \rightarrow 3.13 \cdot 10^{-16}$	15 → 7	$4.02 \cdot 10^{-14} \rightarrow 3.56 \cdot 10^{-16}$	12 → 6

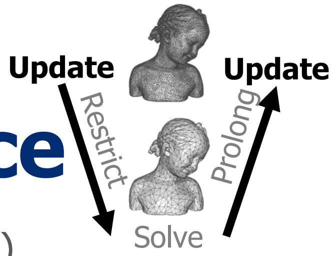
Krylov Subspace Update Convergence

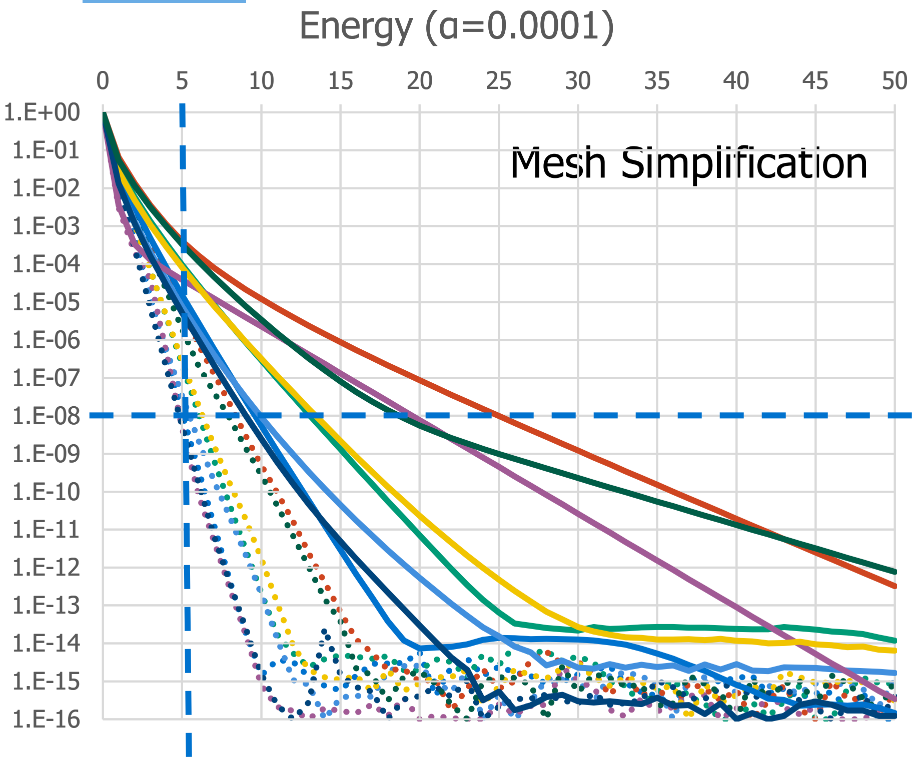


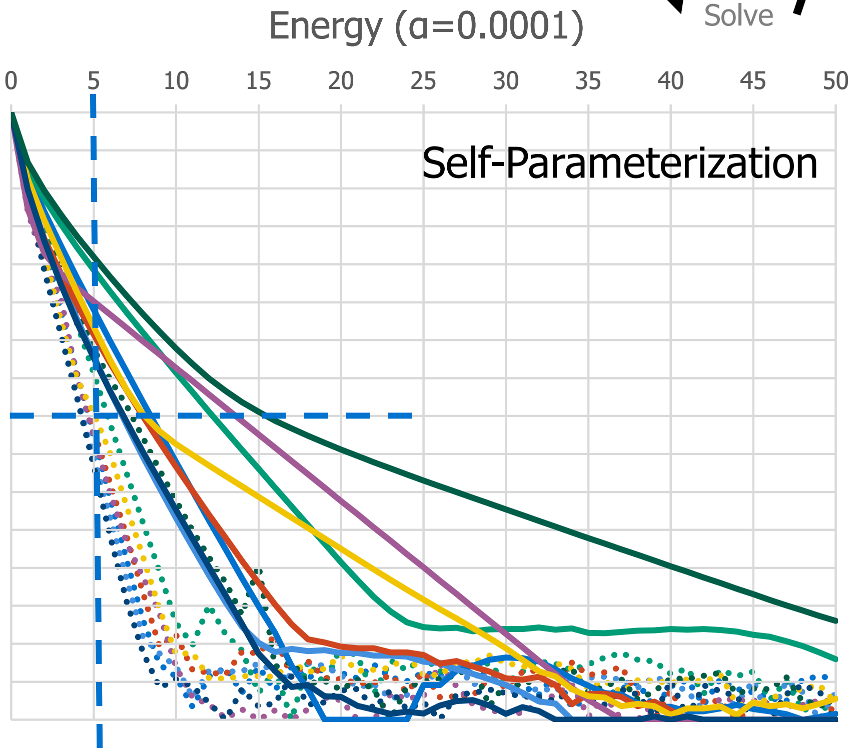
Iterations required to achieve errors $< 10^{-8}$ $(\alpha = 1)$:

Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	**Its $< 1 \cdot 10^{-8}$**	**Err at 50 its.**	**Its $< 1 \cdot 10^{-8}$**
Plane	$1.56 \cdot 10^{-1} \rightarrow 1.34 \cdot 10^{-1}$	>50 → >50	$1.43 \cdot 10^{-1} \rightarrow 1.24 \cdot 10^{-1}$	>50 → >50
Sphere	$1.64 \cdot 10^{-1} \rightarrow 1.36 \cdot 10^{-1}$	>50 → >50	$1.60 \cdot 10^{-1} \rightarrow 1.33 \cdot 10^{-1}$	>50 → >50
Torus	$1.58 \cdot 10^{-1} \rightarrow 1.37 \cdot 10^{-1}$	>50 → >50	$1.46 \cdot 10^{-1} \rightarrow 1.29 \cdot 10^{-1}$	>50 → >50
4-Torus	$4.47 \cdot 10^{-1} \rightarrow 4.35 \cdot 10^{-1}$	>50 → >50	$4.29 \cdot 10^{-1} \rightarrow 4.17 \cdot 10^{-1}$	>50 → >50
Hand	$1.15 \cdot 10^{-1} \rightarrow 7.29 \cdot 10^{-2}$	>50 → >50	$1.09 \cdot 10^{-1} \rightarrow 6.68 \cdot 10^{-2}$	>50 → >50
Bimba	$1.31 \cdot 10^{-1} \rightarrow 1.03 \cdot 10^{-1}$	>50 → >50	$1.07 \cdot 10^{-1} \rightarrow 8.78 \cdot 10^{-2}$	>50 → >50
Rooster	$4.18 \cdot 10^{-1} \rightarrow 4.00 \cdot 10^{-1}$	>50 → >50	$4.14 \cdot 10^{-1} \rightarrow 3.98 \cdot 10^{-1}$	>50 → >50
Fertility	$6.19 \cdot 10^{-1} \rightarrow 6.06 \cdot 10^{-1}$	>50 → >50	$6.15 \cdot 10^{-1} \rightarrow 6.03 \cdot 10^{-1}$	>50 → >50

Krylov Subspace Update Convergence







w/o Krylov   w/ Krylov

SOR + Krylov Subspace Update

Iterations required to achieve errors $< 10^{-8}$ ($\alpha = 0.0001$):

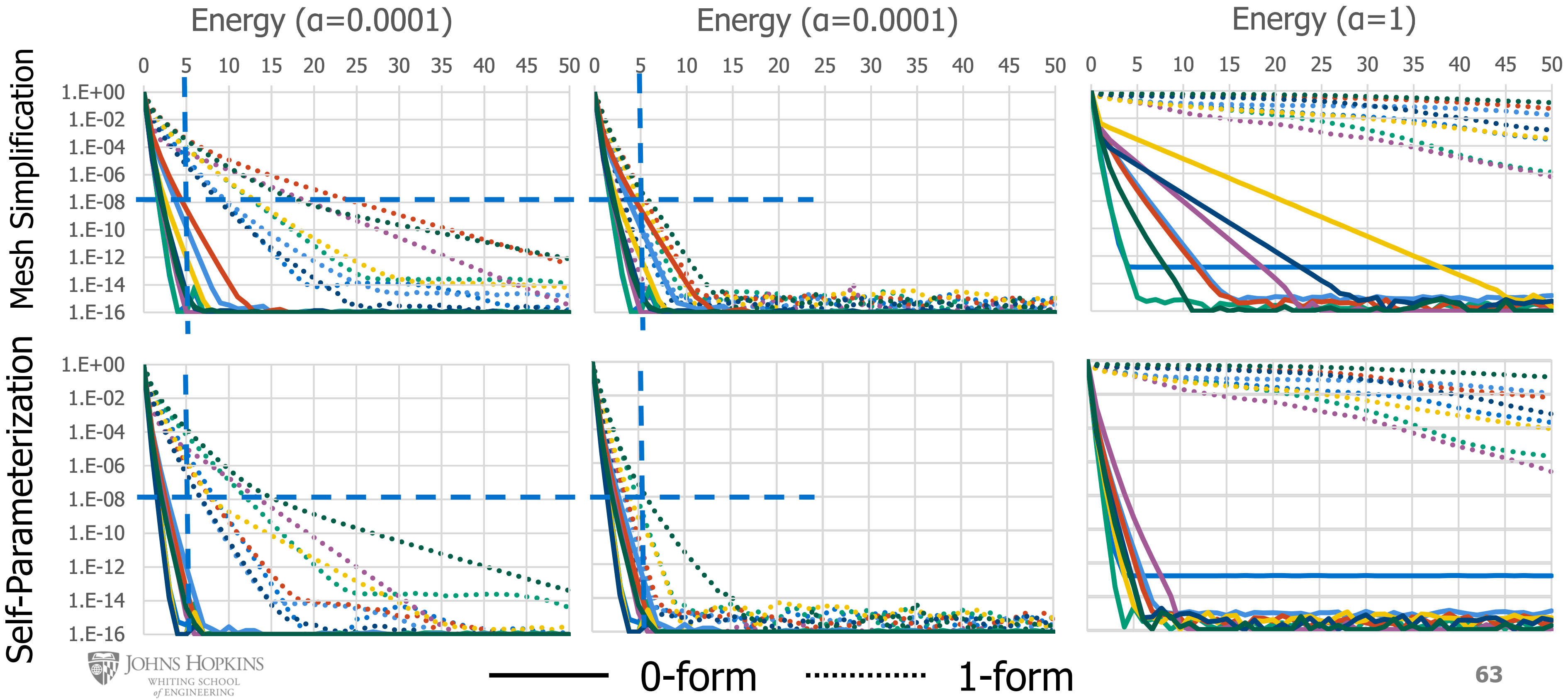
Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	Its $< 1 \cdot 10^{-8}$	Err at 50 its.	Its $< 1 \cdot 10^{-8}$
Plane	$1.45 \cdot 10^{-16} \rightarrow 1.42 \cdot 10^{-15}$	9 → 4	$1.43 \cdot 10^{-16} \rightarrow 2.2 \cdot 10^{-16}$	8 → 4
Sphere	$1.18 \cdot 10^{-14} \rightarrow 1.43 \cdot 10^{-16}$	11 → 5	$2.93 \cdot 10^{-15} \rightarrow 6.2 \cdot 10^{-16}$	11 → 5
Torus	$1.66 \cdot 10^{-15} \rightarrow 3.85 \cdot 10^{-16}$	14 → 4	$1.00 \cdot 10^{-16} \rightarrow 9.51 \cdot 10^{-16}$	6 → 3
4-Torus	$3.22 \cdot 10^{-13} \rightarrow 9.12 \cdot 10^{-16}$	20 → 5	$1.00 \cdot 10^{-16} \rightarrow 5.36 \cdot 10^{-16}$	7 → 4
Hand	$3.42 \cdot 10^{-16} \rightarrow 3.06 \cdot 10^{-16}$	16 → 3	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	11 → 3
Bimba	$6.45 \cdot 10^{-15} \rightarrow 1.06 \cdot 10^{-15}$	12 → 5	$3.54 \cdot 10^{-16} \rightarrow 7.9 \cdot 10^{-16}$	7 → 4
Rooster	$1.21 \cdot 10^{-16} \rightarrow 1.72 \cdot 10^{-16}$	8 → 3	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	6 → 3
Fertility	$7.69 \cdot 10^{-13} \rightarrow 3.75 \cdot 10^{-16}$	15 → 5	$4.02 \cdot 10^{-14} \rightarrow 2.94 \cdot 10^{-16}$	12 → 5

SOR + Krylov Subspace Update

Iterations required to achieve errors $< 10^{-8}$ $(\alpha = 1)$:

Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	Its $< 1 \cdot 10^{-8}$	Err at 50 its.	Its $< 1 \cdot 10^{-8}$
Plane	$1.56 \cdot 10^{-1} \rightarrow \boxed{2.86 \cdot 10^{-4}}$	>50 ➔ >50	$1.43 \cdot 10^{-1} \rightarrow \boxed{2.39 \cdot 10^{-4}}$	>50 ➔ >50
Sphere	$1.64 \cdot 10^{-1} \rightarrow \boxed{1.24 \cdot 10^{-6}}$	>50 ➔ >50	$1.60 \cdot 10^{-1} \rightarrow \boxed{2.45 \cdot 10^{-6}}$	>50 ➔ >50
Torus	$1.58 \cdot 10^{-1} \rightarrow 1.77 \cdot 10^{-2}$	>50 ➔ >50	$1.46 \cdot 10^{-1} \rightarrow 1.30 \cdot 10^{-2}$	>50 ➔ >50
4-Torus	$4.47 \cdot 10^{-1} \rightarrow 5.20 \cdot 10^{-2}$	>50 ➔ >50	$4.29 \cdot 10^{-1} \rightarrow 7.03 \cdot 10^{-3}$	>50 ➔ >50
Hand	$1.15 \cdot 10^{-1} \rightarrow \boxed{5.65 \cdot 10^{-7}}$	>50 ➔ >50	$1.09 \cdot 10^{-1} \rightarrow \boxed{2.90 \cdot 10^{-7}}$	>50 ➔ >50
Bimba	$1.31 \cdot 10^{-1} \rightarrow \boxed{3.32 \cdot 10^{-4}}$	>50 ➔ >50	$1.07 \cdot 10^{-1} \rightarrow \boxed{1.03 \cdot 10^{-4}}$	>50 ➔ >50
Rooster	$4.18 \cdot 10^{-1} \rightarrow 1.43 \cdot 10^{-3}$	>50 ➔ >50	$4.14 \cdot 10^{-1} \rightarrow \boxed{7.73 \cdot 10^{-4}}$	>50 ➔ >50
Fertility	$6.19 \cdot 10^{-1} \rightarrow 1.40 \cdot 10^{-1}$	>50 ➔ >50	$6.15 \cdot 10^{-1} \rightarrow 1.19 \cdot 10^{-1}$	>50 ➔ >50

0-Form vs 1-Form (SOR + Krylov)



All

Iterations required to achieve errors $< 10^{-8}$ ($\alpha = 0.0001$):

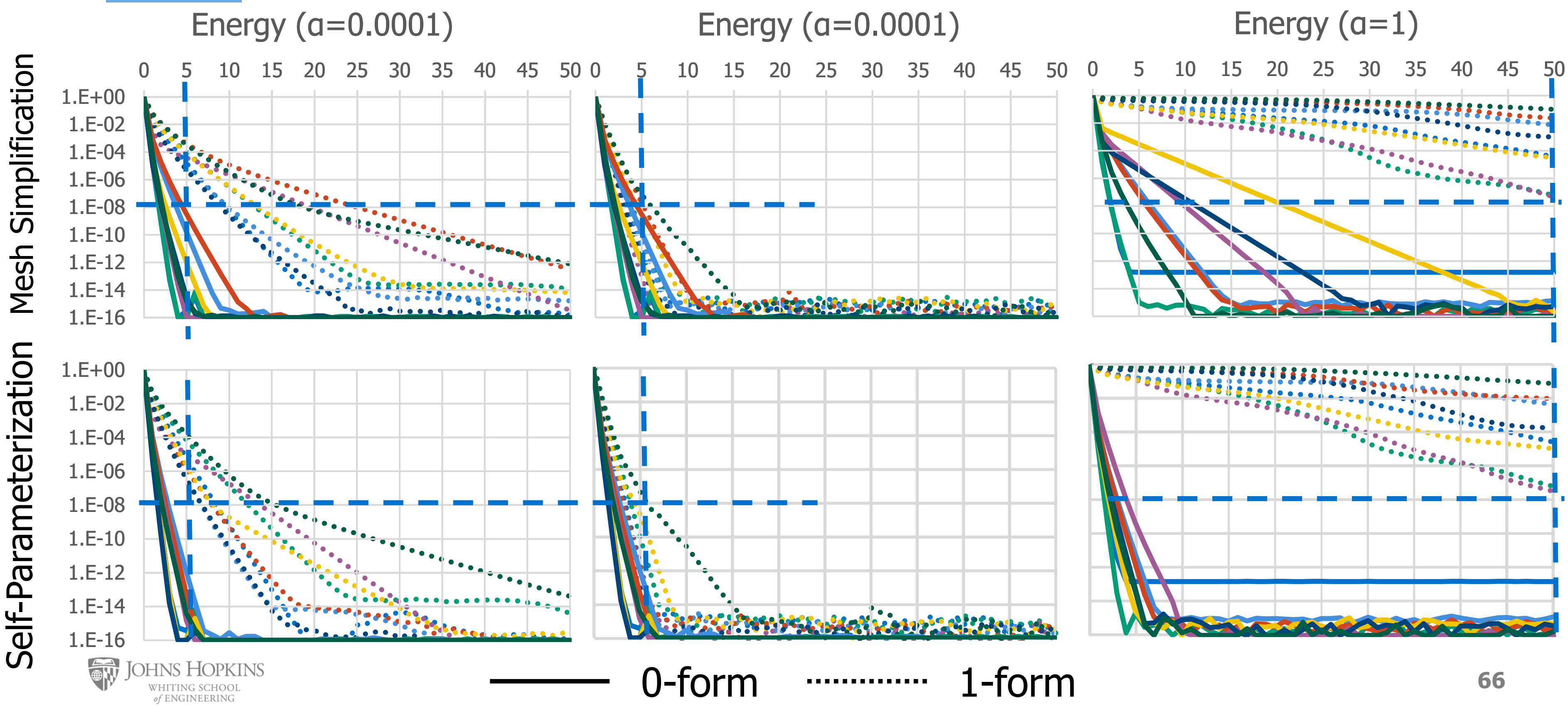
Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	Its $< 1 \cdot 10^{-8}$	Err at 50 its.	Its $< 1 \cdot 10^{-8}$
Plane	$1.45 \cdot 10^{-16} \rightarrow 1.04 \cdot 10^{-15}$	9 → 4	$1.43 \cdot 10^{-16} \rightarrow 3.11 \cdot 10^{-16}$	8 → 3
Sphere	$1.18 \cdot 10^{-14} \rightarrow 4.51 \cdot 10^{-16}$	11 → 4	$2.93 \cdot 10^{-15} \rightarrow 2.19 \cdot 10^{-16}$	11 → 4
Torus	$1.66 \cdot 10^{-15} \rightarrow 3.98 \cdot 10^{-16}$	14 → 4	$1.00 \cdot 10^{-16} \rightarrow 1.40 \cdot 10^{-16}$	6 → 3
4-Torus	$3.22 \cdot 10^{-13} \rightarrow 4.80 \cdot 10^{-16}$	20 → 5	$1.00 \cdot 10^{-16} \rightarrow 5.67 \cdot 10^{-16}$	7 → 4
Hand	$3.42 \cdot 10^{-16} \rightarrow 1.07 \cdot 10^{-16}$	16 → 3	$1.00 \cdot 10^{-16} \rightarrow 1.00 \cdot 10^{-16}$	11 → 3
Bimba	$6.45 \cdot 10^{-15} \rightarrow 7.47 \cdot 10^{-16}$	12 → 4	$3.54 \cdot 10^{-16} \rightarrow 2.88 \cdot 10^{-16}$	7 → 4
Rooster	$1.21 \cdot 10^{-16} \rightarrow 3.73 \cdot 10^{-16}$	8 → 3	$1.00 \cdot 10^{-16} \rightarrow 2.99 \cdot 10^{-16}$	6 → 3
Fertility	$7.69 \cdot 10^{-13} \rightarrow 1.00 \cdot 10^{-16}$	15 → 5	$4.02 \cdot 10^{-14} \rightarrow 4.16 \cdot 10^{-16}$	12 → 5

All

Iterations required to achieve errors $< 10^{-8}$ $(\alpha = 1)$:

Models	Mesh Simplification		Self-Parameterization	
	Err at 50 its.	Its $< 1 \cdot 10^{-8}$	Err at 50 its.	Its $< 1 \cdot 10^{-8}$
Plane	$1.56 \cdot 10^{-1} \rightarrow 3.61 \cdot 10^{-5}$	>50 ➔ >50	$1.43 \cdot 10^{-1} \rightarrow 2.40 \cdot 10^{-5}$	>50 ➔ >50
Sphere	$1.64 \cdot 10^{-1} \rightarrow 5.60 \cdot 10^{-8}$	>50 ➔ >49	$1.60 \cdot 10^{-1} \rightarrow 5.77 \cdot 10^{-8}$	>50 ➔ 49
Torus	$1.58 \cdot 10^{-1} \rightarrow 7.51 \cdot 10^{-3}$	>50 ➔ >50	$1.46 \cdot 10^{-1} \rightarrow 4.18 \cdot 10^{-3}$	>50 ➔ >50
4-Torus	$4.47 \cdot 10^{-1} \rightarrow 2.35 \cdot 10^{-2}$	>50 ➔ >50	$4.29 \cdot 10^{-1} \rightarrow 8.89 \cdot 10^{-3}$	>50 ➔ >50
Hand	$1.15 \cdot 10^{-1} \rightarrow 4.40 \cdot 10^{-8}$	>50 ➔ >49	$1.09 \cdot 10^{-1} \rightarrow 2.93 \cdot 10^{-8}$	>50 ➔ >47
Bimba	$1.31 \cdot 10^{-1} \rightarrow 3.14 \cdot 10^{-5}$	>50 ➔ >50	$1.07 \cdot 10^{-1} \rightarrow 9.59 \cdot 10^{-6}$	>50 ➔ >50
Rooster	$4.18 \cdot 10^{-1} \rightarrow 1.00 \cdot 10^{-3}$	>50 ➔ >50	$4.14 \cdot 10^{-1} \rightarrow 1.65 \cdot 10^{-4}$	>50 ➔ >50
Fertility	$6.19 \cdot 10^{-1} \rightarrow 1.02 \cdot 10^{-1}$	>50 ➔ >50	$6.15 \cdot 10^{-1} \rightarrow 6.78 \cdot 10^{-2}$	>50 ➔ >50

0-Form vs 1-Form (All)



Multigrid – Prolongation Matrix

- Ideal multigrid:
  - Relaxation solves high-frequency
  - Coarse level solves low-frequency

- Ideal prolongation Matrix:
  - Coarse functions should be low-frequency

Analyzing Prolongation Matrices

Idea:

- Check that functions that "appear" low-frequency in the coarse space are low-frequency in the fine space

Implementation:

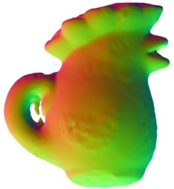
- Solve (Laplacian) eigenvalue problem at each level
- Project the coarse eigenfunctions $\{\hat{\mathbf{e}}_i\}$ onto the finest eigenfunctions $\{\mathbf{e}_\mathbf{j}\}$:

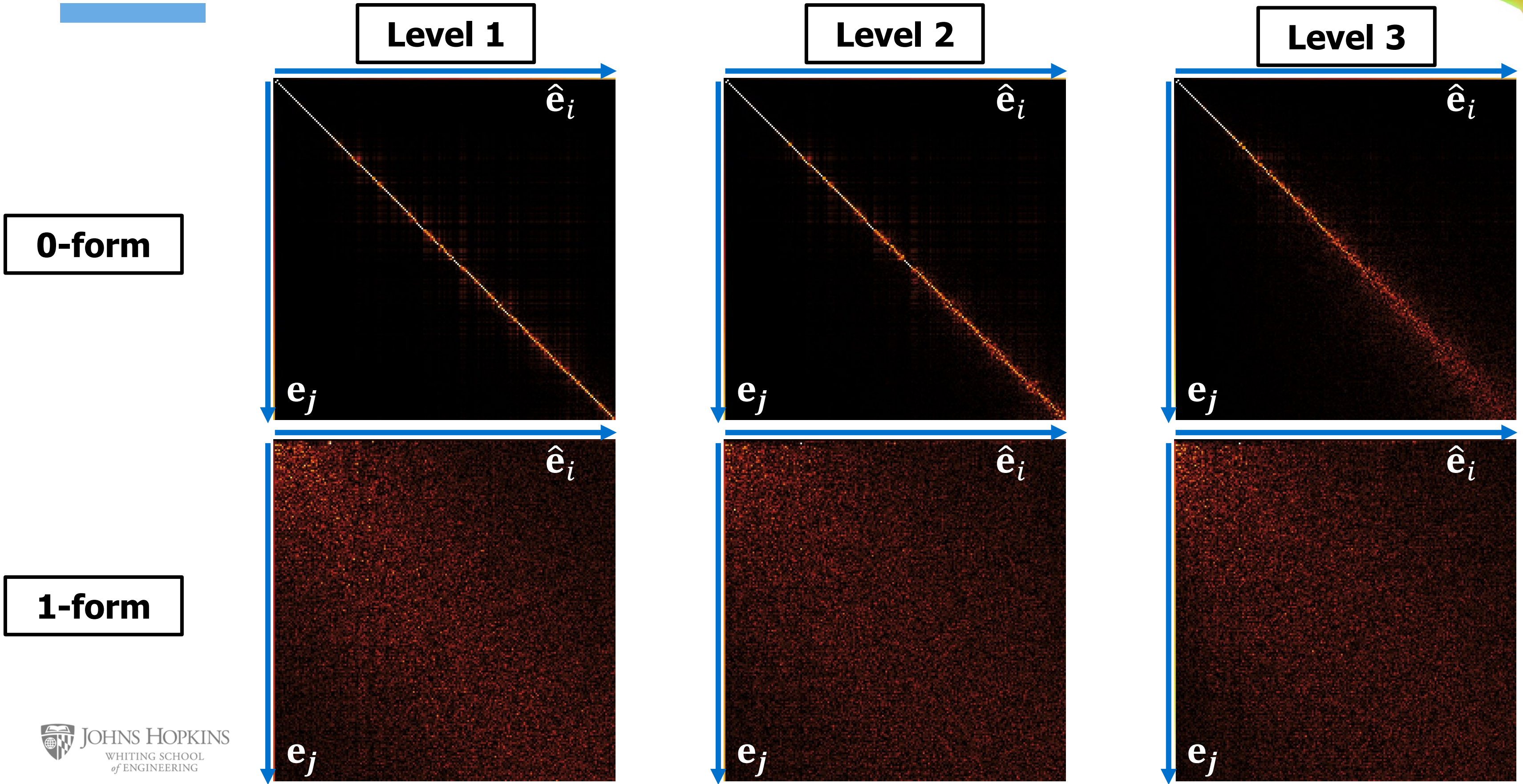
$$C_{ij} = \langle \hat{\mathbf{e}}_i, \mathbf{e}_j \rangle$$

Expectation:

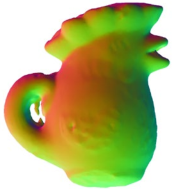
- For smaller values of (i, j), the matrix C should look diagonal

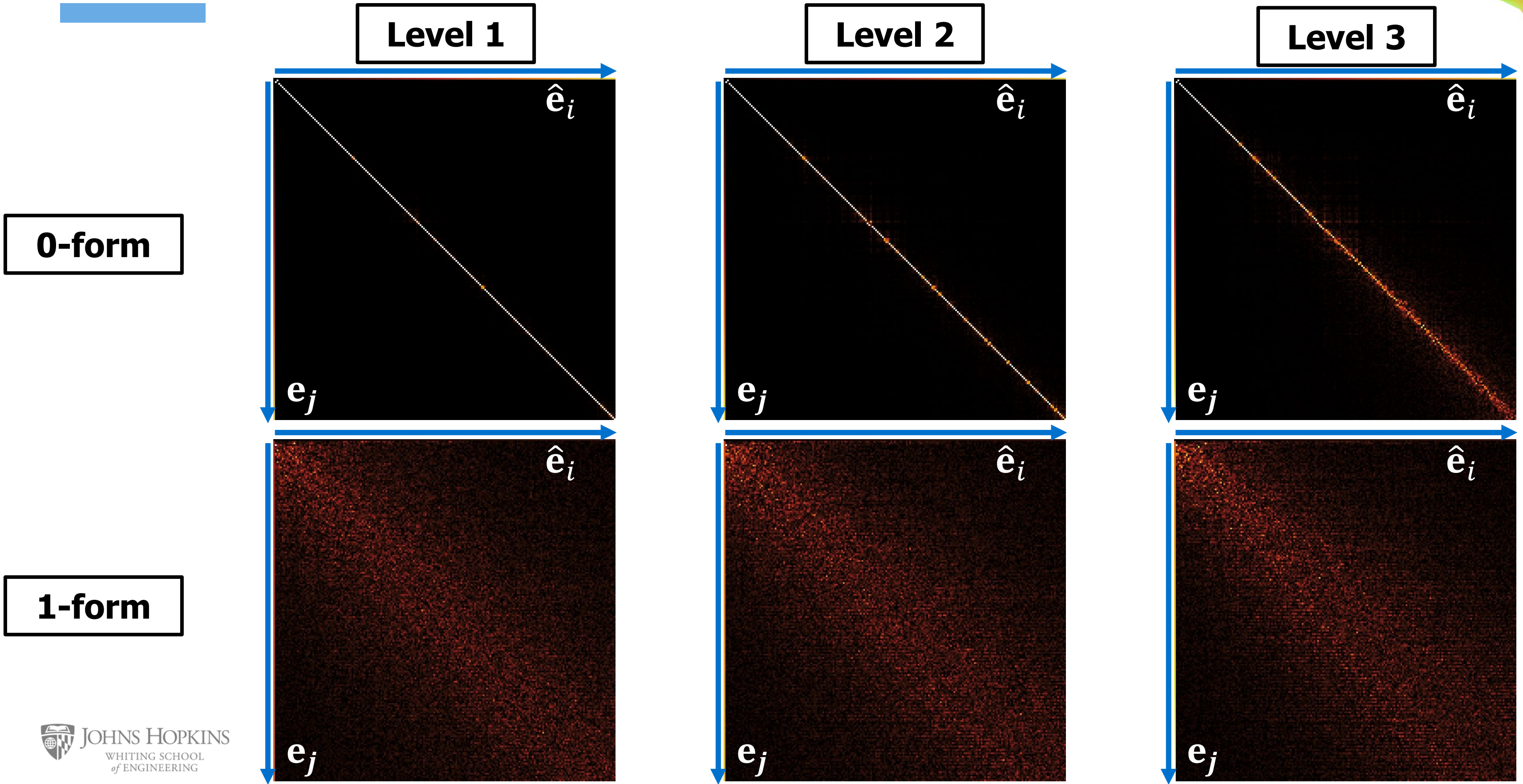
Mesh Simplification C (Rooster)



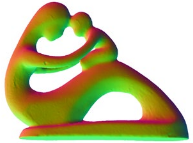


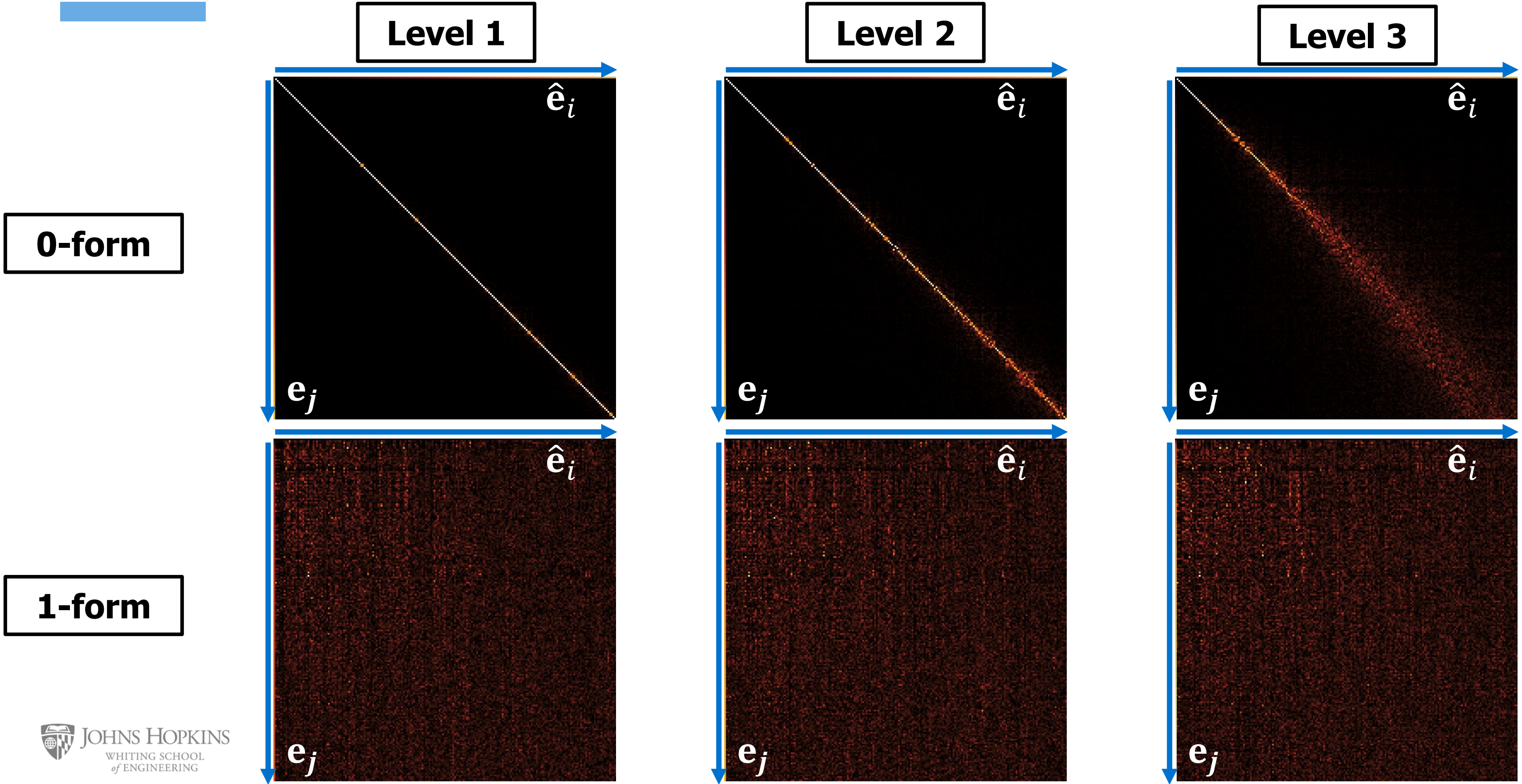
Self-Parameterization C (Rooster)



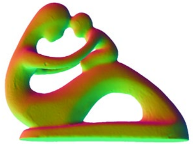


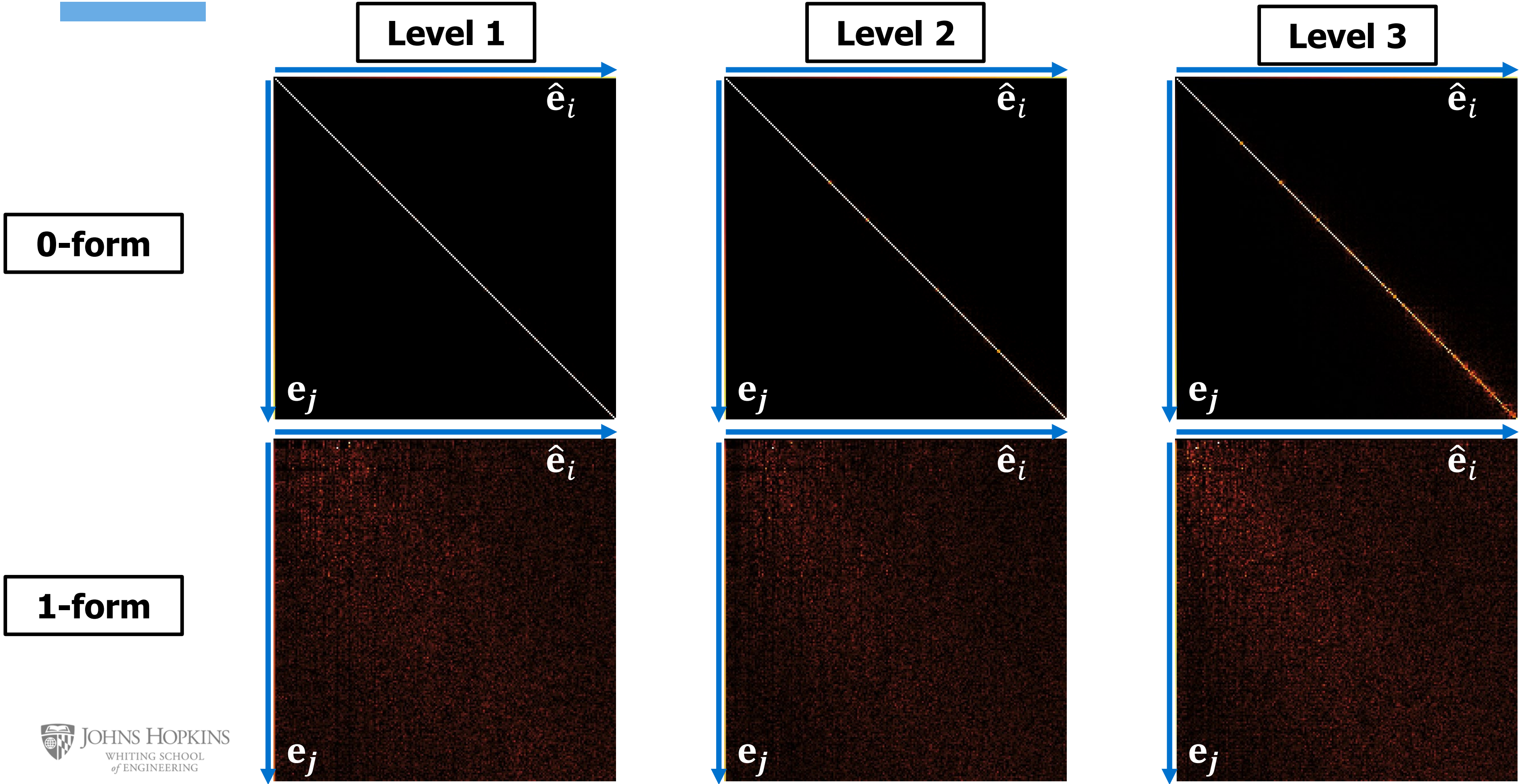
Mesh Simplification C (Fertility)





Self-Parameterization C (Fertility)





Outline

Hierarchical Gradient Domain Vector Field Processing

1. Why are we interested in **vector field processing**?
2. How do we perform vector field processing in the **gradient domain**?
3. How do we design a **hierarchy** for vector field processing?
4. How do we make the hierarchical solver **efficient**?
5. Conclusion

Conclusion

1. Why are we interested in (hierarchical) **vector field processing**?

- This thesis is the first to look into a hierarchical approach for vector field processing
  - Wide range applications
  - Real-time performance

- Applicable to improve the performance of the earlier work finding shape correspondences, where optical flow is used [S. C. Lee, M. Kazhdan. SGP 2019]

Conclusion

2. How do we perform vector field processing in the **gradient domain**?

- Formulate as a gradient domain problem using exterior derivatives
- Discretize using a basis, resulting in solving a linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

- We propose to use any 1-form basis that is constructed from a 0-form basis for the discretization

Conclusion

3. How do we design a **hierarchy** for vector field processing?

- We show that the multi-linear formulation allows us to naturally extend the 0-form prolongation to a 1-form prolongation
  - "Naturally" ⇒ commutes with differentiation

- We propose to use this 1-form prolongation for the multigrid method

- Our generalization can be applied to other 0-form-based-constructed hierarchy, such as the 2-form hierarchy in our work (just accepted today) [M. Kohlbrenner, S. C. Lee, M. Kazhdan, M. Alexa. SGP 2023].

Conclusion

4. How do we make the hierarchical solver **converge** quickly?

Challenge:

- Slow convergence

Solutions:

- Successive over-relaxation
- Prolongation smoothing
- **Krylov subspace update**

Future Work

Remaining Challenges:

- Slow convergence for big α
- Coarse basis functions do not “focus” on the low-frequency

Possible Directions:

- Pre-conditioning
- Hierarchy (mesh simplification) customized for 1-form prolongation
- Modified 0-form prolongation matrix

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  - Nassir and all the previous collaborators for the interesting work on
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    [M. Fischer, B. Fuerst, S. C. Lee, J. Fotouhi, S. Habert, S. Weidert, E. Euler, G. Osgood, N. Navab, IJCAR 2016]
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    [S. C. Lee, M. Seibold, P. Furnstahl M. Farshad, N. Navab. SPIE 2020]
    [M. Haiderbhai, S. Ledesma, S. C. Lee, M. Seibold, P. Furnstahl, N. Navab, P. Fallavollita. IJCAR 2020]
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