# Optical Flow

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# Overview

- Optical flow on the image plane ($\mathcal{I} \subset \mathbb{R}^2$)
- Optical flow on the sphere ($S^2$)
- Optical flow on a 2D surface ($\mathcal{M}$)

# Optical flow – images

- Given two intensity functions on the image plane $\mathcal{I} \subset \mathbb{R}^2$

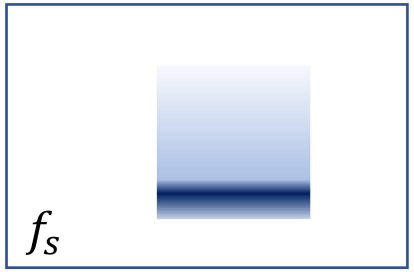
$$f_s, f_t : \mathcal{I} \to \mathbb{R}$$

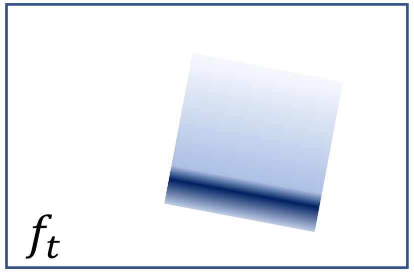
- An optical flow is a vector field $\vec{v} \in \Gamma(\mathrm{T}\mathcal{I})$ such that

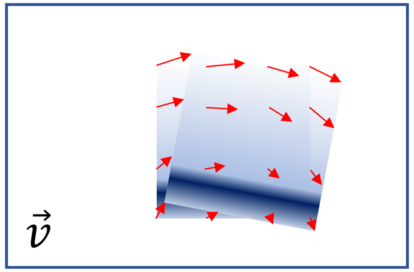
$$f_s(p) = f_t(p + \vec{v}(p))$$

Where $\mathrm{T}\mathcal{I} = \cup_p \mathrm{T}_p\mathcal{I}$ (the tangent bundle of $\mathcal{I}$)

- Optical flow describes the "apparent" motion between two images
    - Brightness (signal) constancy assumption



# Optical flow – formulation

$$E(\vec{v}) = \int_{\mathcal{I}} \left(f_s(p) - f_t(p + \vec{v}(p))\right)^2 dp$$

- Using 1st order Taylor expansion:

$$f(p + \vec{v}(p)) \approx f(p) + \vec{v}(p) \cdot \nabla f(p)$$

- Denoting $\delta(p) = f_s(p) - f_t(p)$, the energy becomes

$$E(\vec{v}) \approx \int_{\mathcal{I}} \left(\delta(p) - \vec{v}(p) \ \cdot \nabla f_t(p)\right)^2 dp$$

# Optical flow – least square solution

$$E(\vec{v}) = \int_{\mathcal{J}} \left(\delta(p) - \vec{v}(p) \ \cdot \nabla f_t(p)\right)^2 dp$$

- Least square solution:

$$A\vec{v} = b$$

where

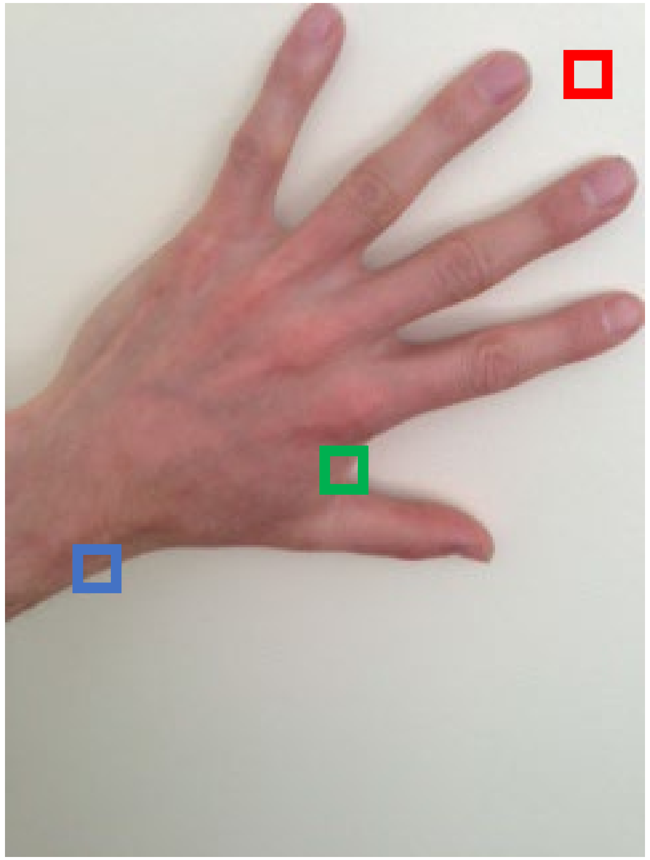
$$A = \nabla f_t \otimes \nabla f_t, \qquad b_i = \delta \nabla f_t$$

- Only require knowing the image difference $\delta$ and the gradient $\nabla f_t$

# Optical flow – ill-conditioned

$$A = \nabla f_t \otimes \nabla f_t$$

- Depends solely on the image gradients
- $A$ is at most rank 1:
  - Edges (changes in one dominant direction)
- It can be rank 0:
  - Corners / intensity changes in different directions
  - Uniform regions (small gradients)
- Need to regularize the system



# Optical flow – small motion constraint

- Assume the motion is small

$$E(\vec{v}) = \int_{\mathcal{J}} \left(\delta(p) - \vec{v}(p) \ \cdot \nabla f_t(p)\right)^2 dp + \epsilon \int_{\mathcal{J}} \|\vec{v}(p)\|^2 dp$$

- Numerically, this adds a regularizer to the system (without changing the right-hand-side):

$$A = \nabla f_t \otimes \nabla f_t + \epsilon I$$

- $A$ is now s.p.d. and invertible.

# Optical flow – coarse-to-fine refinement

- In practice, refinement is required, because
  - Using 1st order approximation
  - The motion between the two images is not small
  - The solution often is a ***local minima***
- Gaussian image pyramid
  - Image is blurred and down-sampled such that pixel changes represent a larger motion
    - $f^{i+1} = D(G * f^{i})$
  - Coarser level results are up-sampled to warp the finer source image to target
    - $f^{i} = f^{i+1}(p + U\vec{v}_s^{i+1}(p))$
  - Result: a composition of vector fields

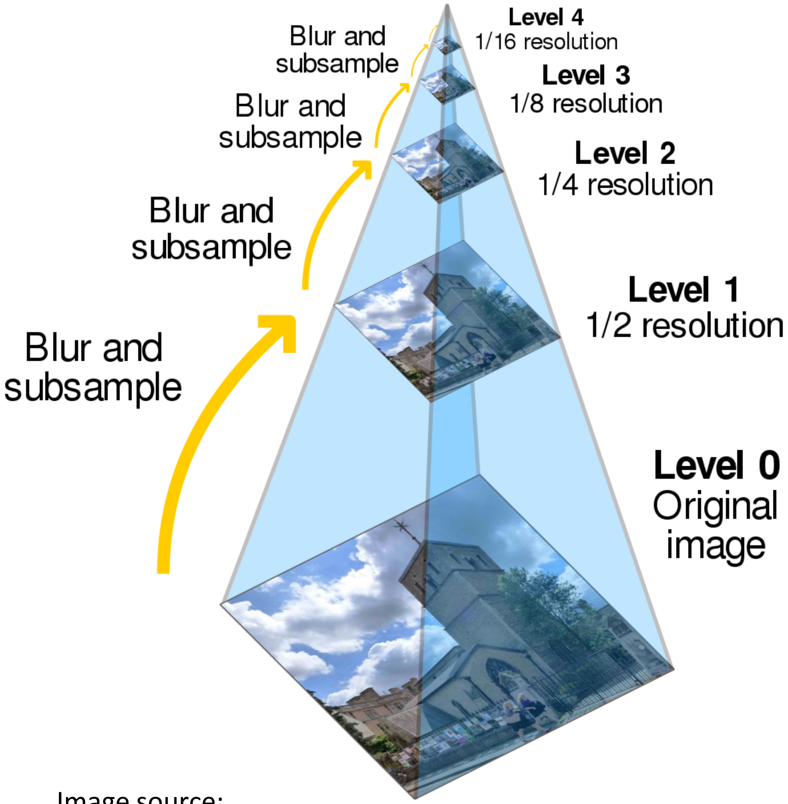



Image source:
https://en.wikipedia.org/wiki/Pyramid_(image_processing)

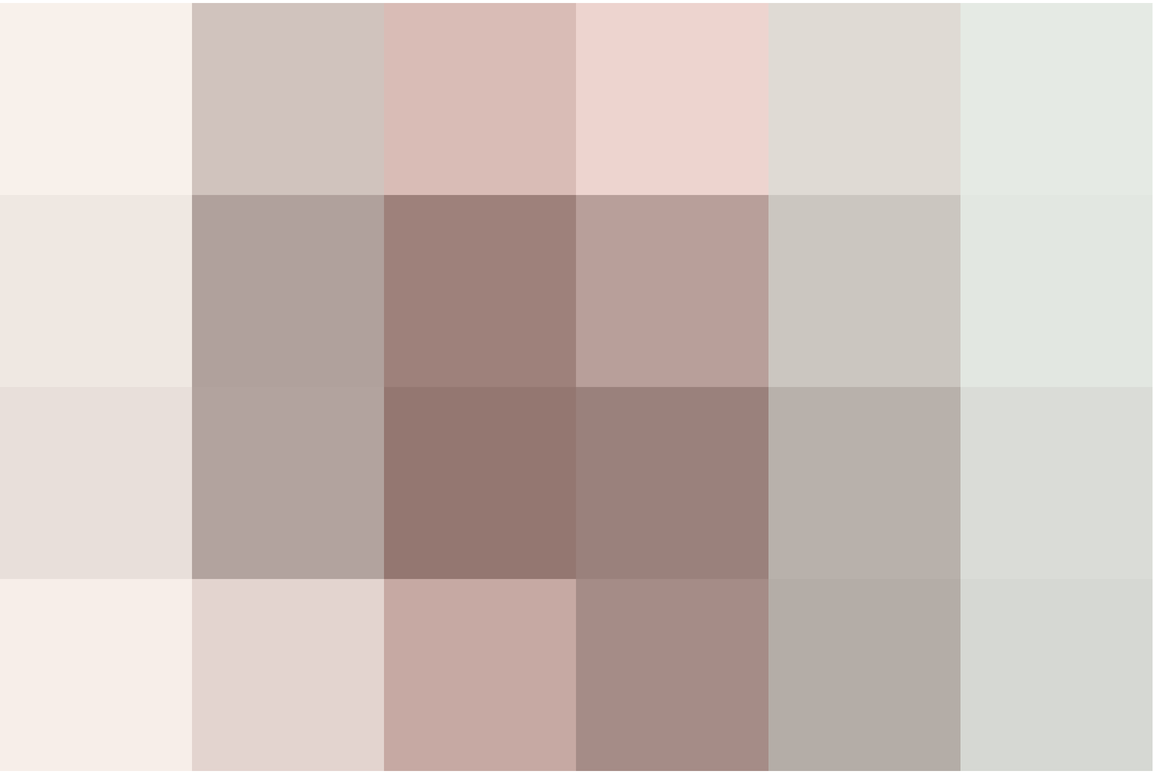
# Application – image registration input

**Source**

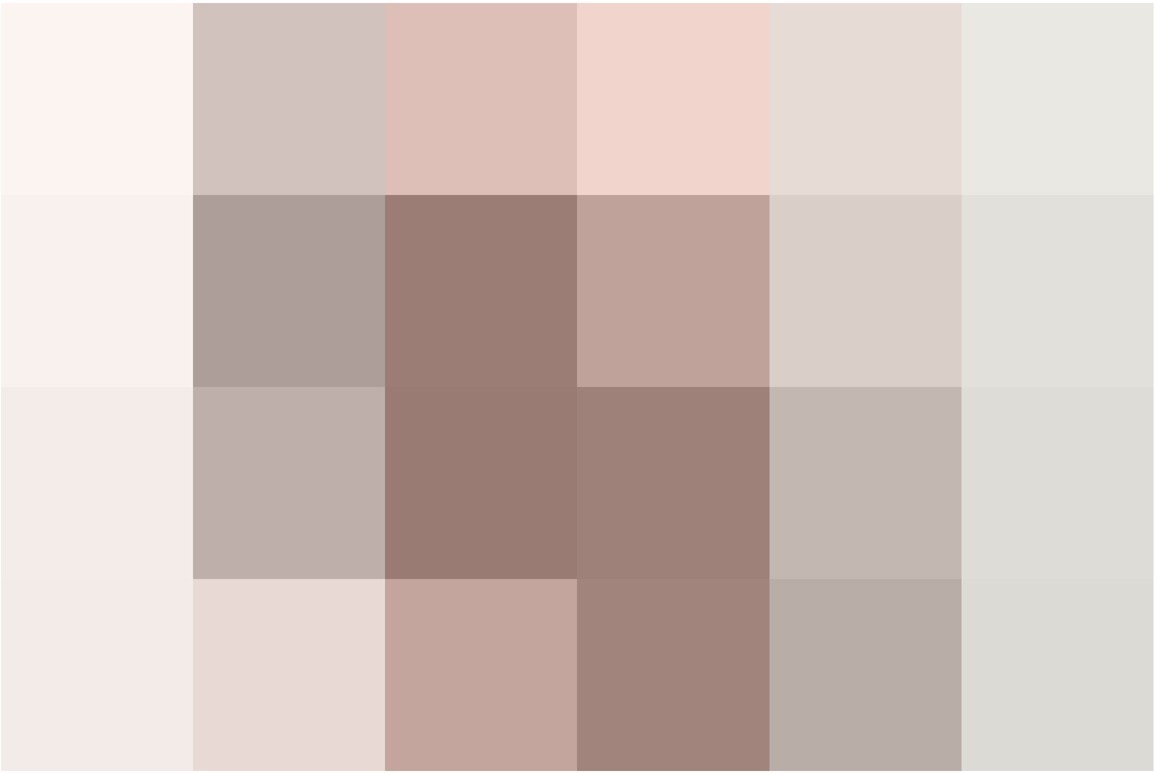
**Target**

# Application – image registration level 6

**Source**

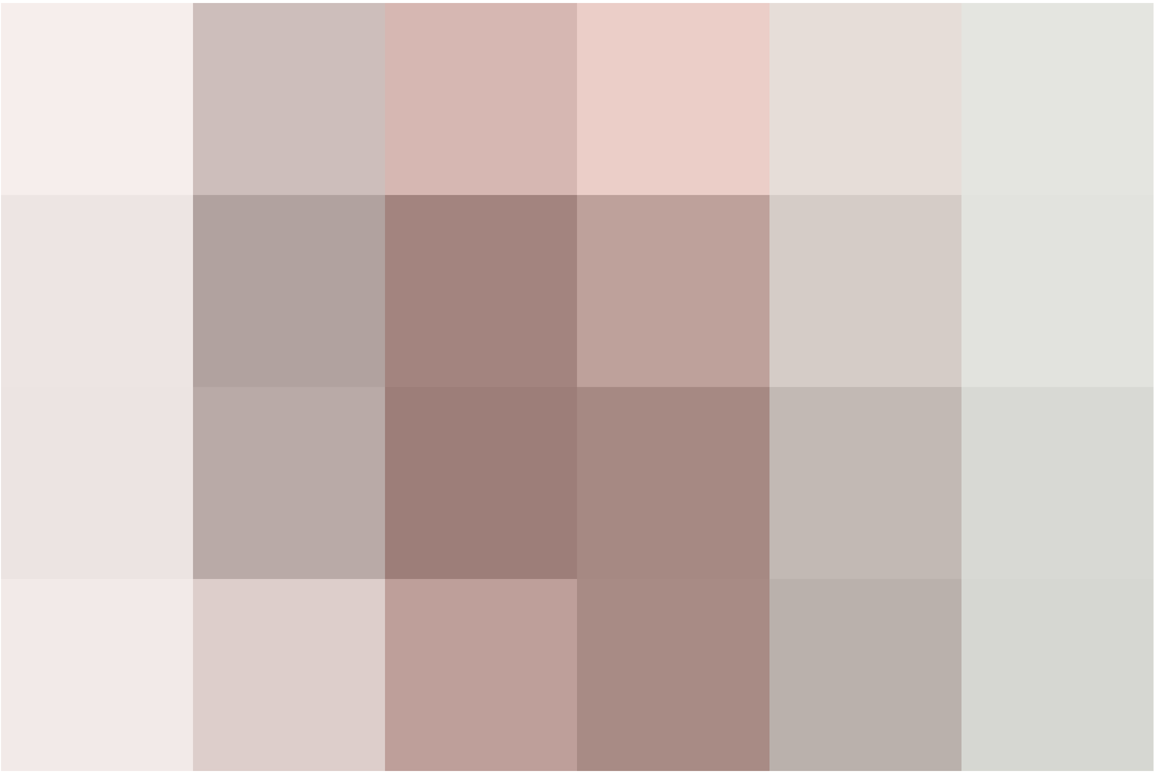


**Target**

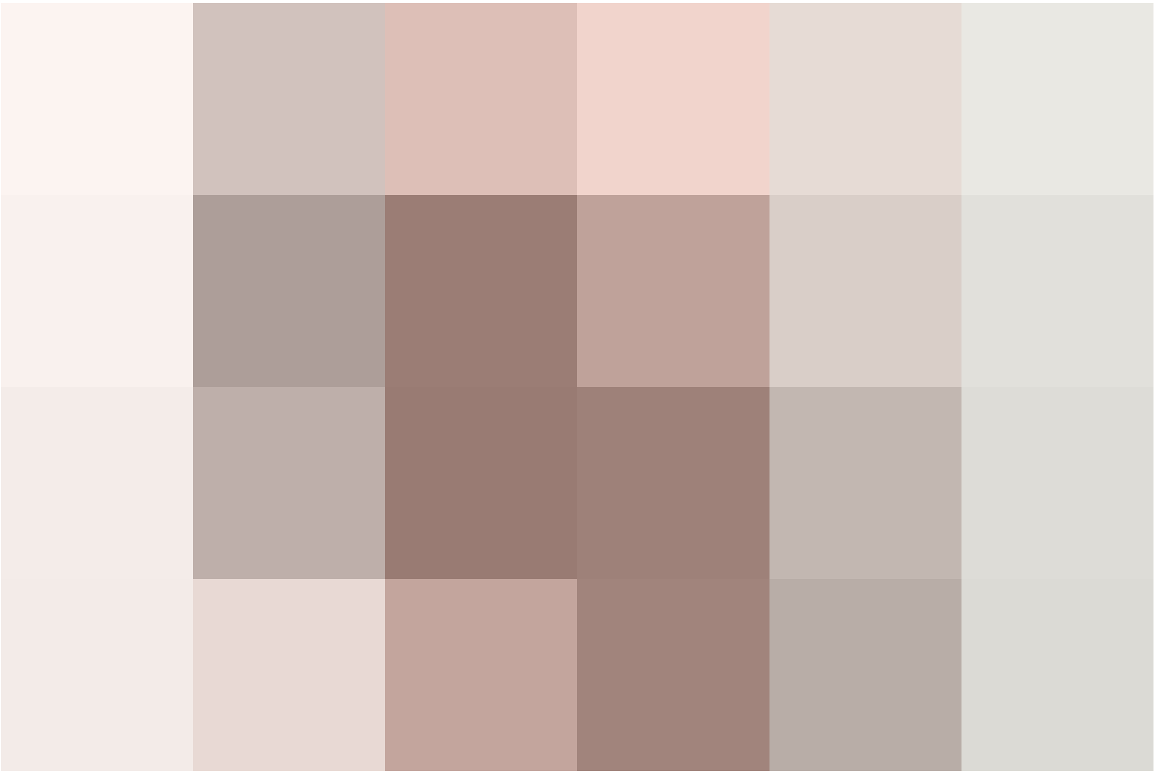


# Application – image registration level 6

**Source to target**

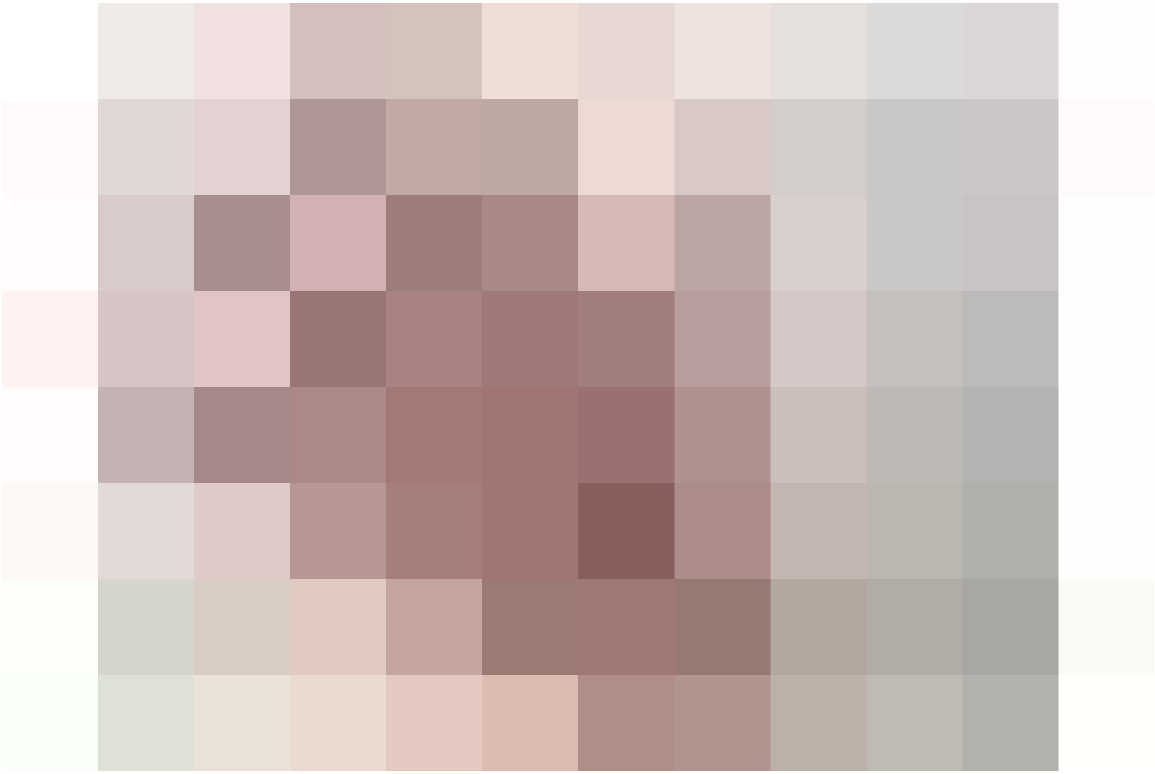


**Target**

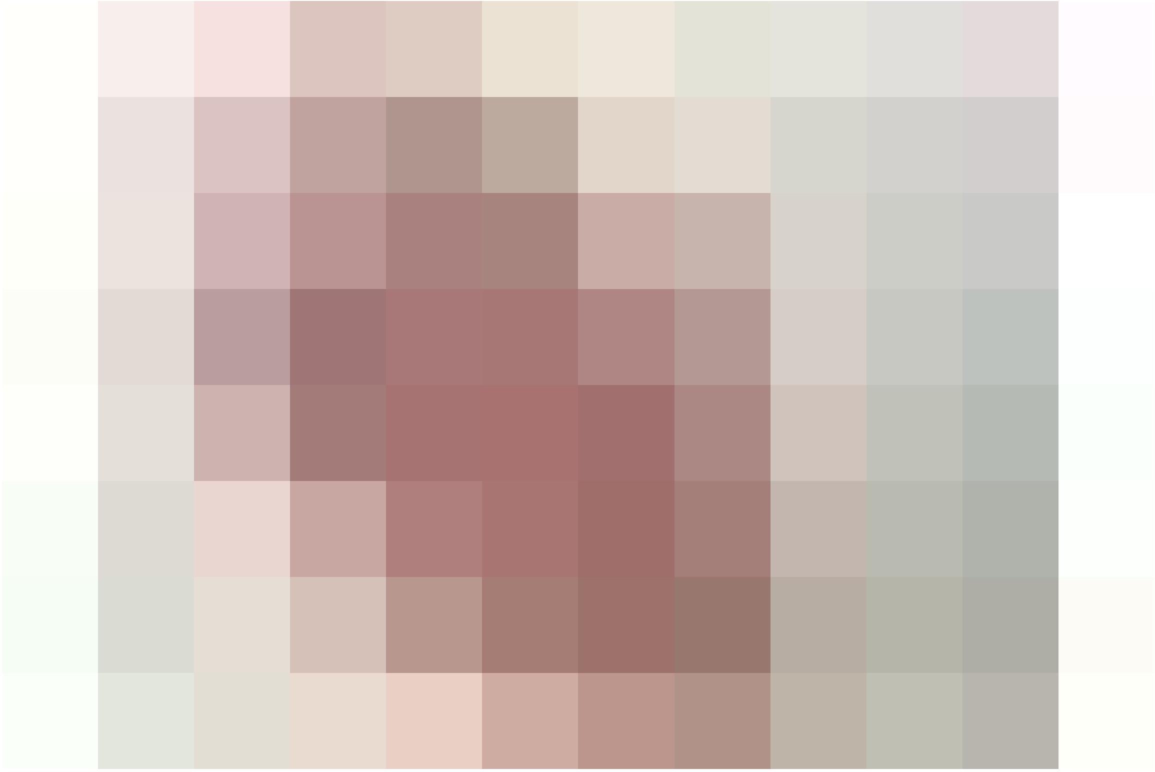


# Application – image registration level 5

**Source**

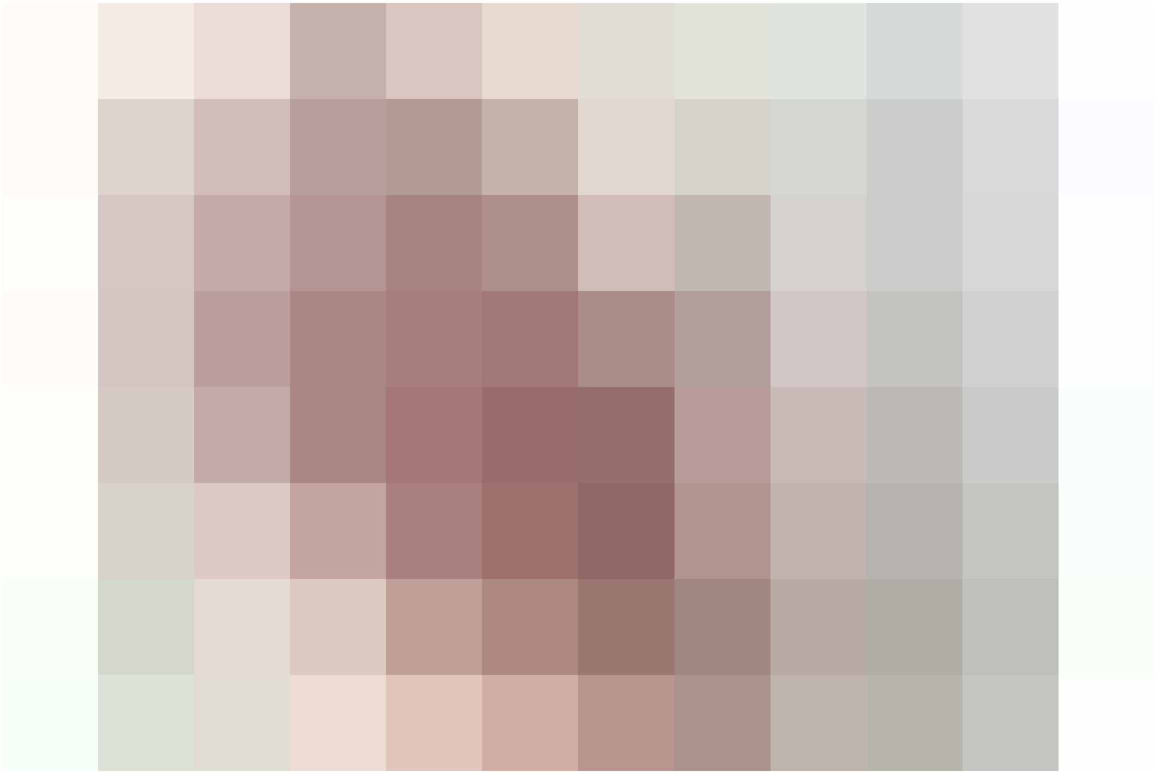


**Target**

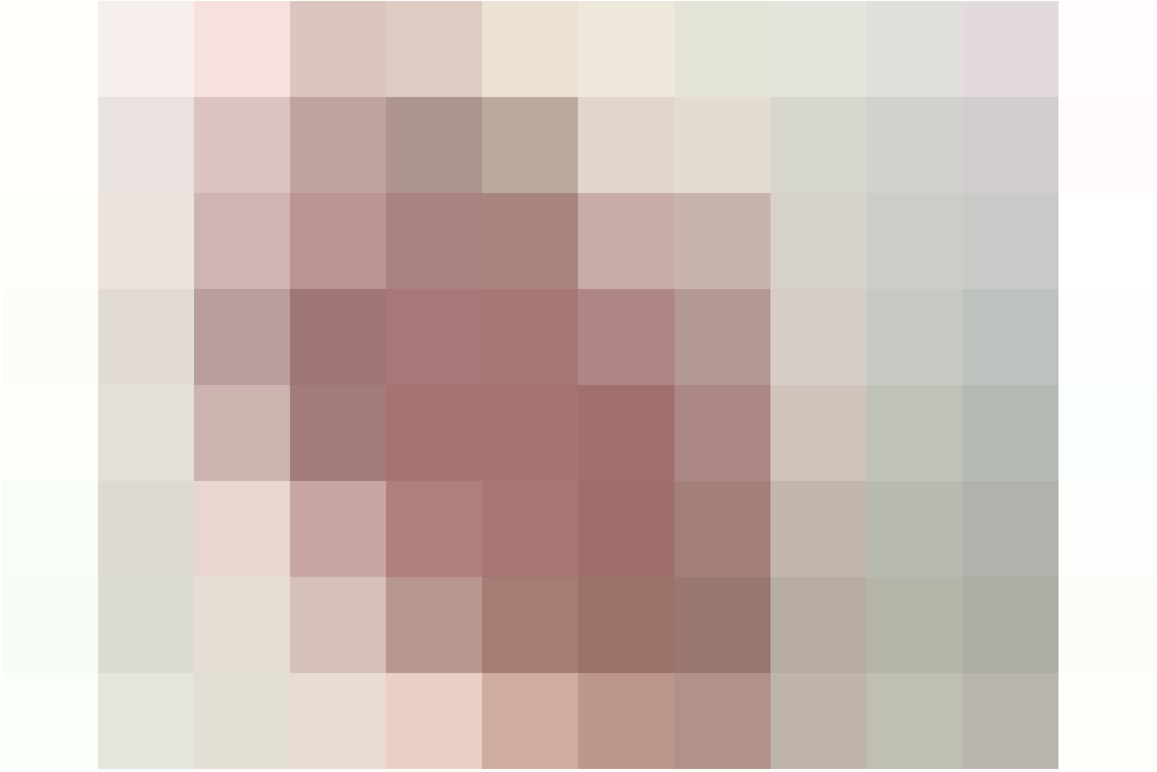


# Application – image registration level 5

**Source to target upsampled**

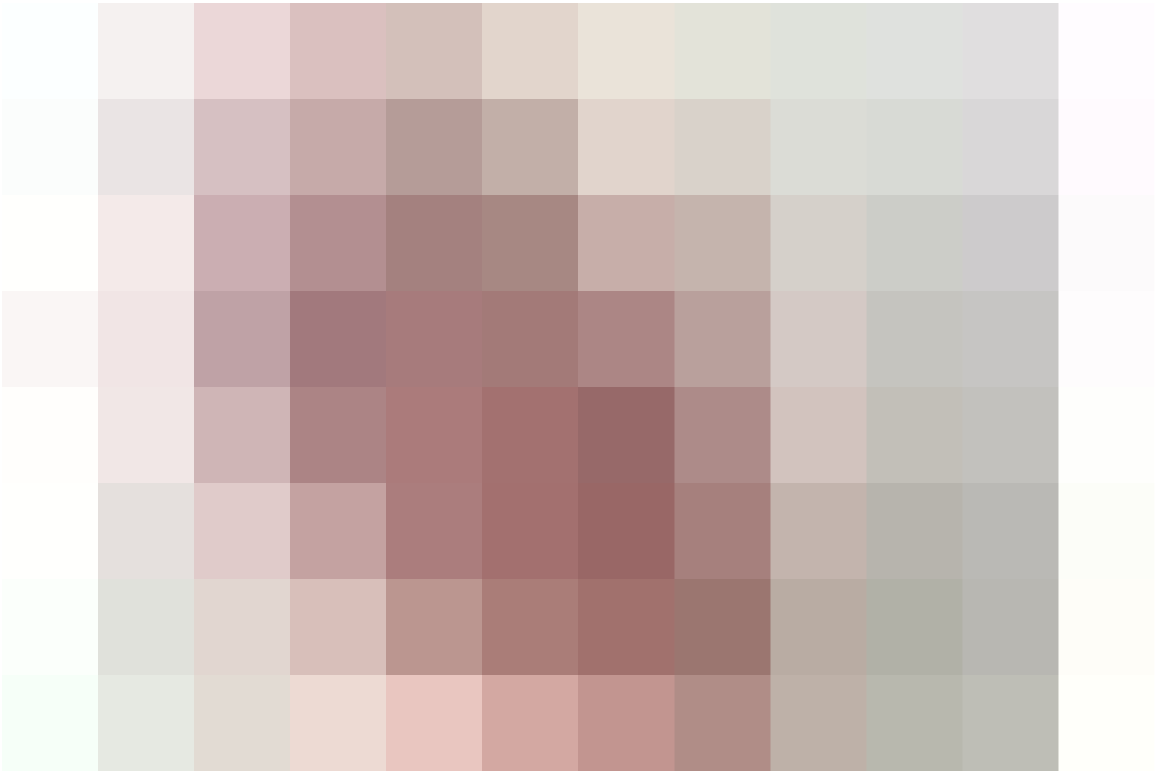


**Target**

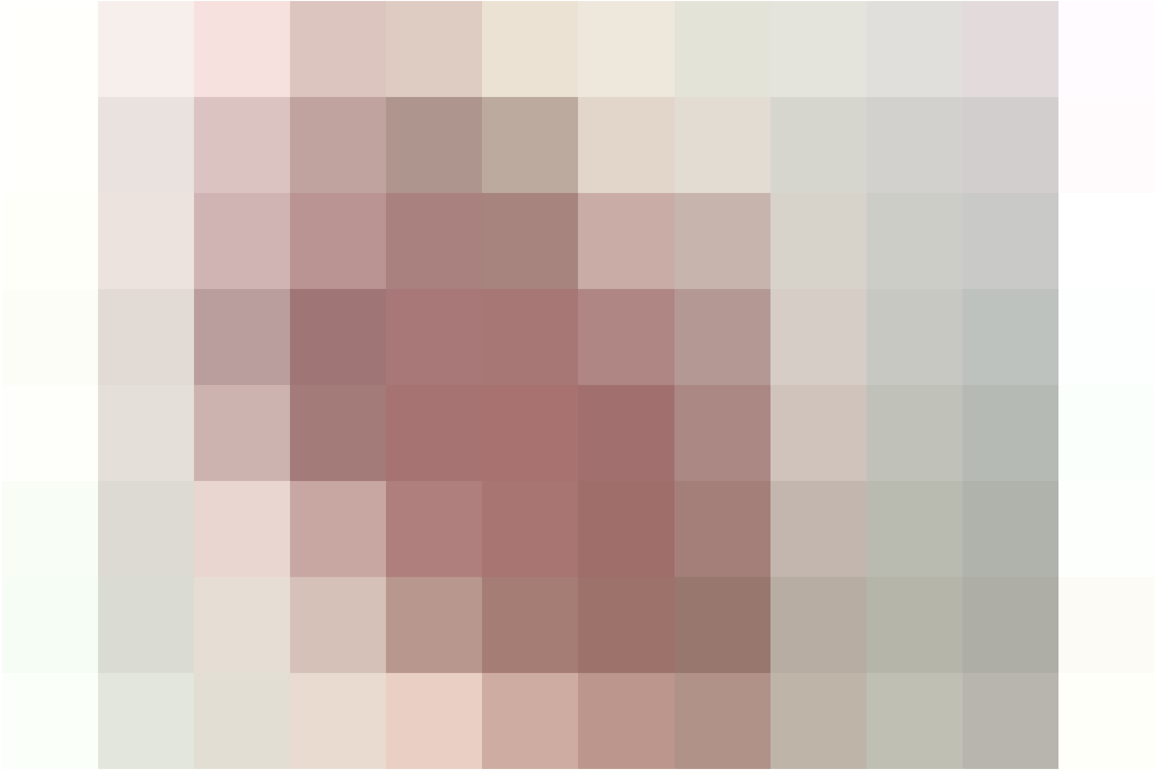


# Application – image registration level 5

**Source to target**

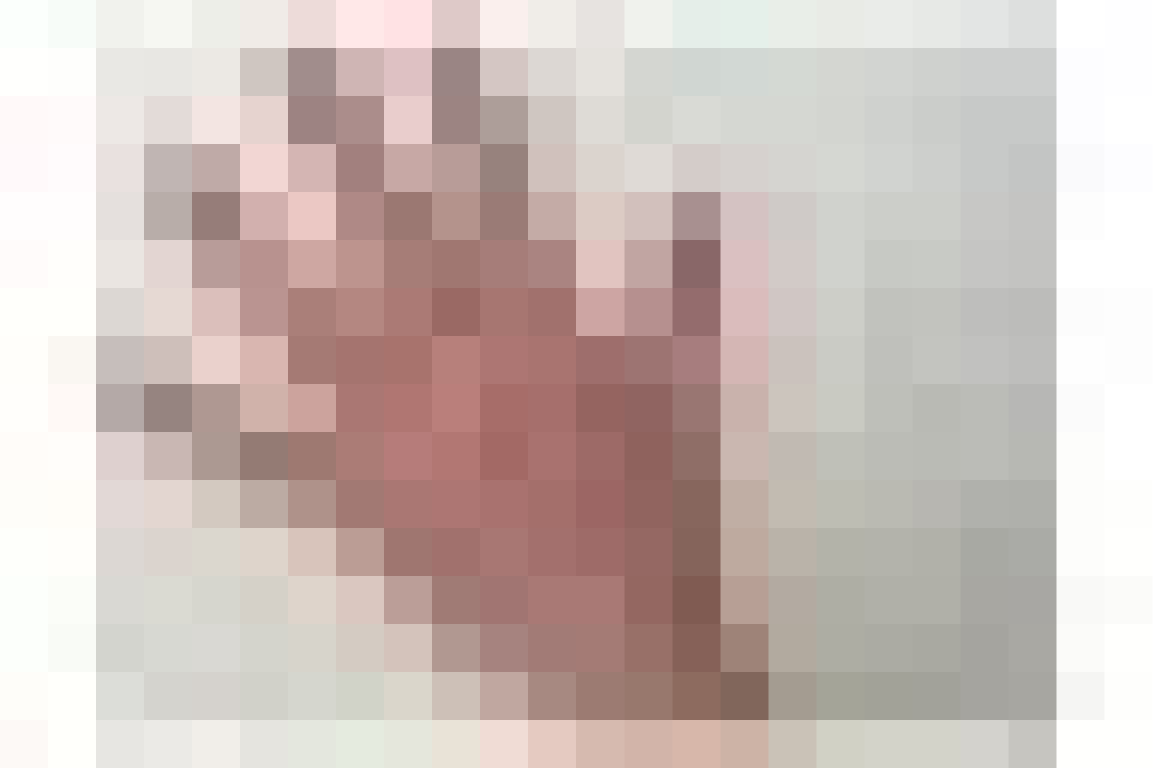


**Target**

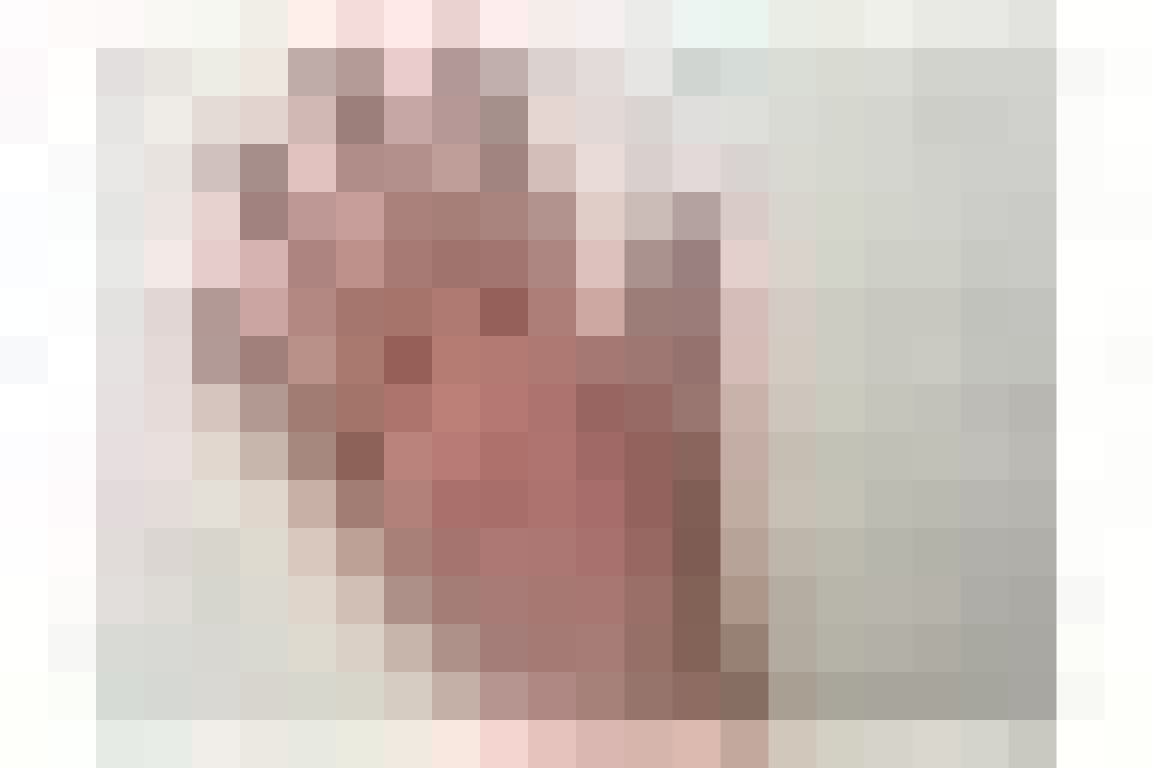


# Application – image registration level 4

**Source**



**Target**



# Application – image registration level 4

**Source to target upsampled**

**Target**

# Application – image registration level 4

**Source to target**

**Target**

# Application – image registration level 3

**Source**

**Target**

# Application – image registration level 3

**Source to target upsampled**

**Target**

# Application – image registration level 3

**Source to target**

**Target**

# Application – image registration level 2

**Source**

**Target**

# Application – image registration level 2

**Source to target upsampled**

**Target**

# Application – image registration level 2

**Source to target**

**Target**

# Application – image registration level 1

**Source**

**Target**

# Application – image registration level 1

**Source to target upsampled**

**Target**

# Application – image registration level 1

**Source to target**

**Target**

# Application – image registration level 0

**Source**

**Target**

# Application – image registration level 0

**Source to target upsampled**

**Target**

# Application – image registration level 0

**Source to target**

**Target**

# Application – image registration output

**Source to target**

**Target**

# Optical flow – images

- Simple formulation (signal constancy):

$$E(\vec{v}) = \int_{\mathcal{J}} \big(\delta(p) - \vec{v}(p) \ \cdot \nabla f_t(p)\big)^2 \, dp$$

- Regularization (small motion constraint):

$$\epsilon \int_{\mathcal{J}} \|\vec{v}(p)\|^2 dp$$

- Coarse-to-fine hierarchy (image pyramid):

# Overview

- Optical flow on the image plane ($\mathcal{I} \subset \mathbb{R}^2$)
- Optical flow on the sphere ($S^2$)
- Optical flow on a 2D surface ($\mathcal{M}$)

# Optical flow – spherical images

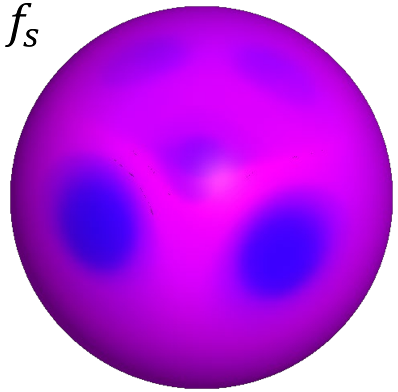
- Given two functions on the sphere $\mathcal{S}^2 \subset \mathbb{R}^3$

$$f_s, f_t : \mathcal{S}^2 \to \mathbb{R}$$

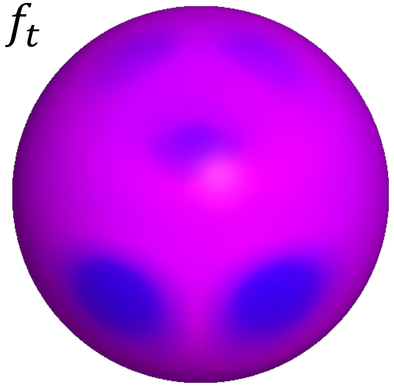
- An optical flow is a vector field $\vec{v} \in \Gamma(\mathrm{T}\mathcal{S}^2)$ such that

$$f_s(p) = f_t\left(\exp_p \vec{v}(p)\right)$$

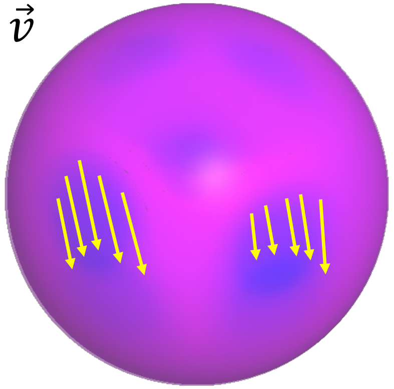
$f_s$



$f_t$



$\vec{v}$



# Optical flow – formulation

$$E(\vec{v}) = \int_{\mathcal{S}^2} \left( f_s(p) - f_t\left(\exp_p \vec{v}(p)\right)\right)^2 dp$$

- Using 1st order Taylor expansion:

$$f\left(\exp_p \vec{v}(p)\right) \approx f(p) + \langle \vec{v}(p), \nabla f(p) \rangle$$

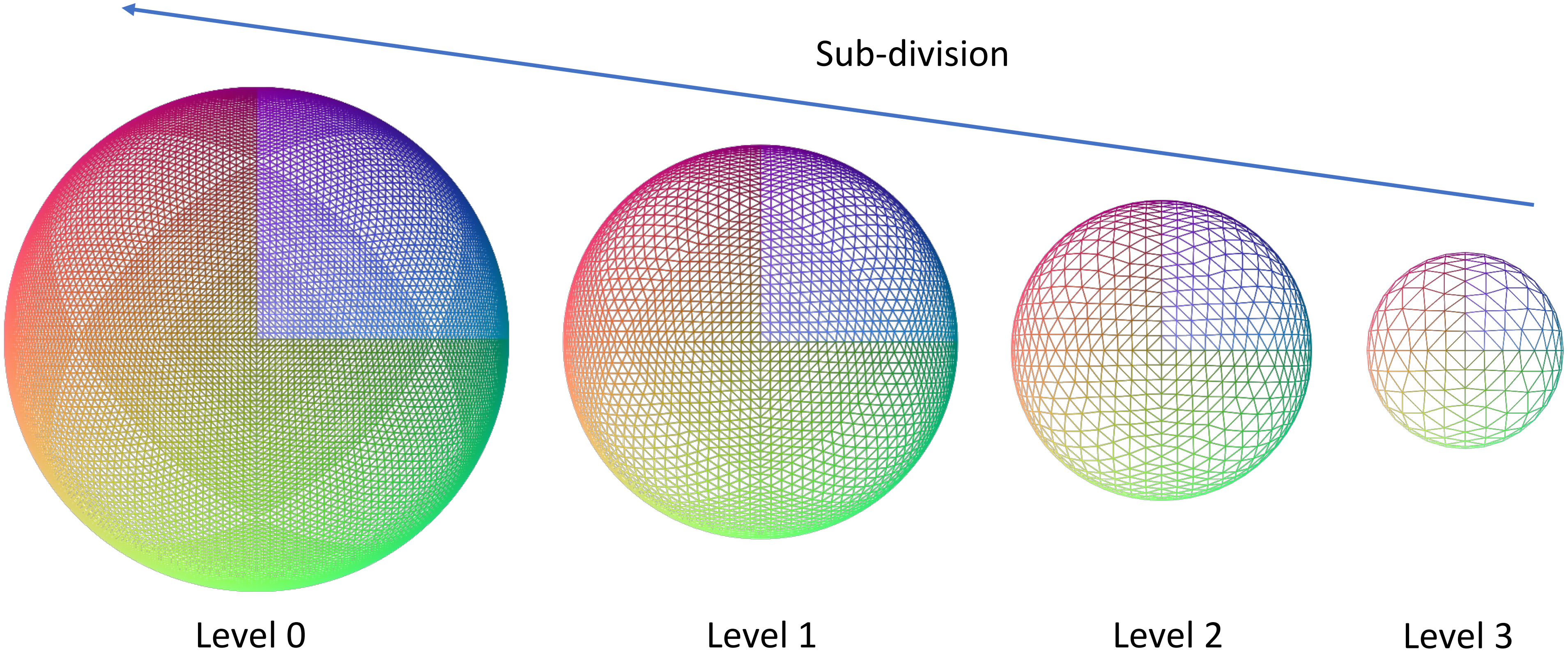
where$\langle \cdot,\cdot \rangle$ is an inner product: $\mathrm{T}_p\mathcal{S}^2 \times \mathrm{T}_p\mathcal{S}^2 \to \mathbb{R}$

- Denote $\delta(p) = f_s(p) - f_t(p)$, the energy becomes

$$E(\vec{v}) \approx \int_{\mathcal{S}^2} (\delta - \langle \vec{v}, \nabla f_t \rangle)^2 \, dp$$

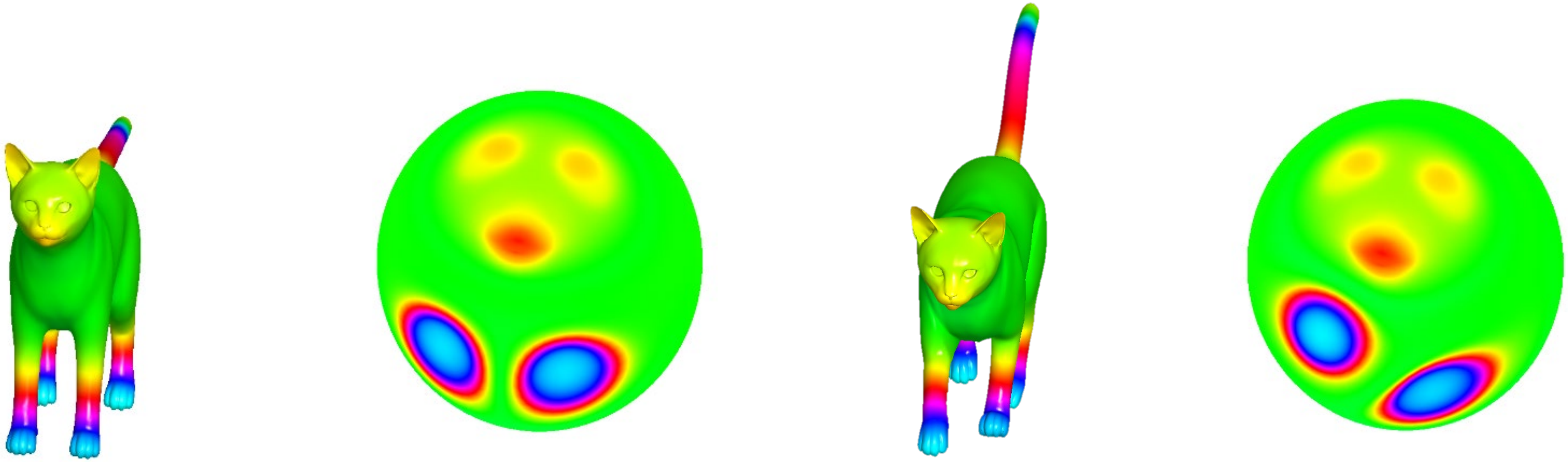
# Optical flow – coarse-to-fine refinement

- Can use sub-division to create the coarse-to-fine hierarchy



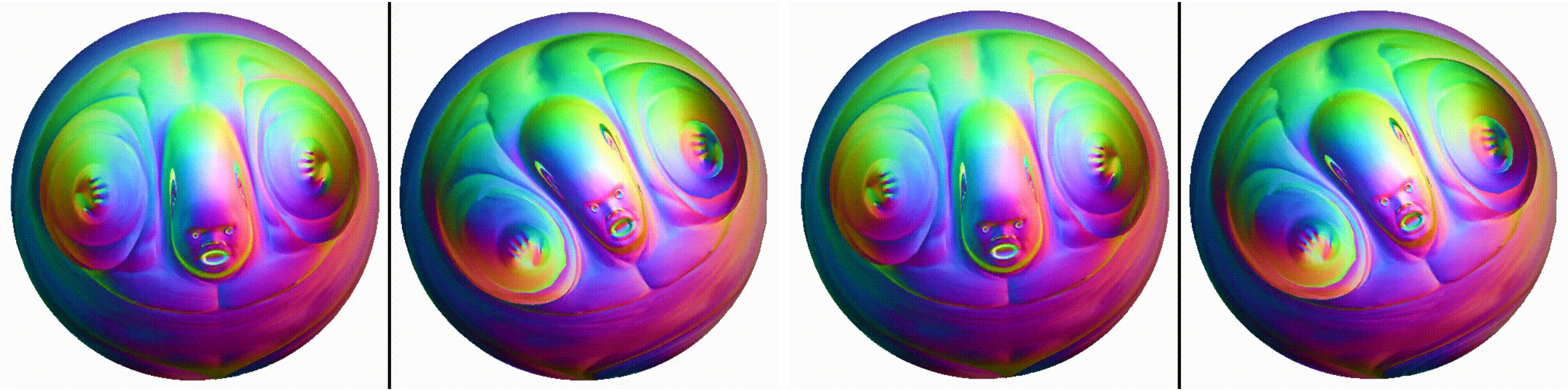
# Application – shape registration

- We parameterize (genus-zero) shapes over the sphere and compute spherical functions pulling back geometric information

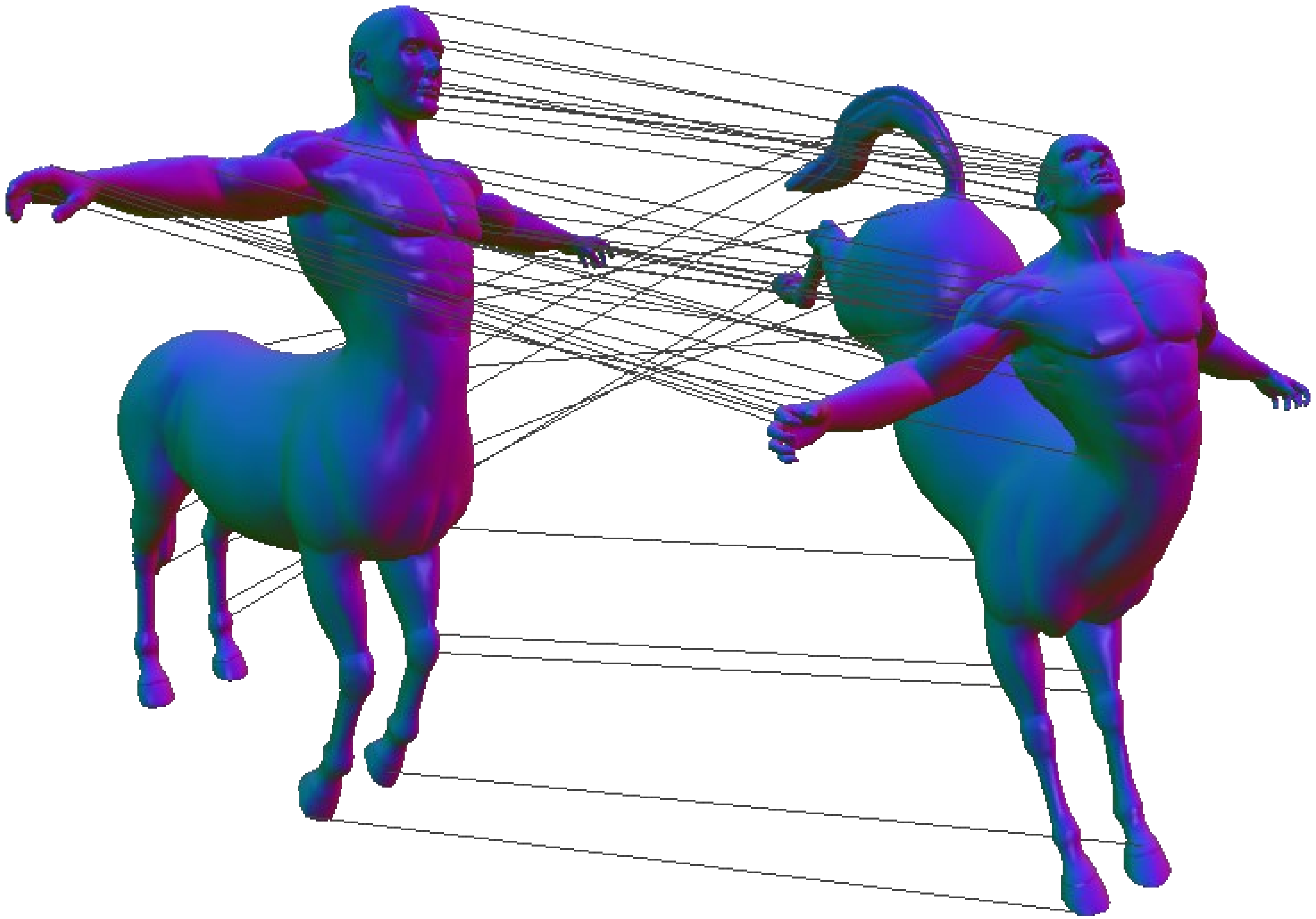


# Application – shape registration

- After an initial (Möbius) alignment, refine using optical flow



# Application – shape registration



# Optical flow – spherical images

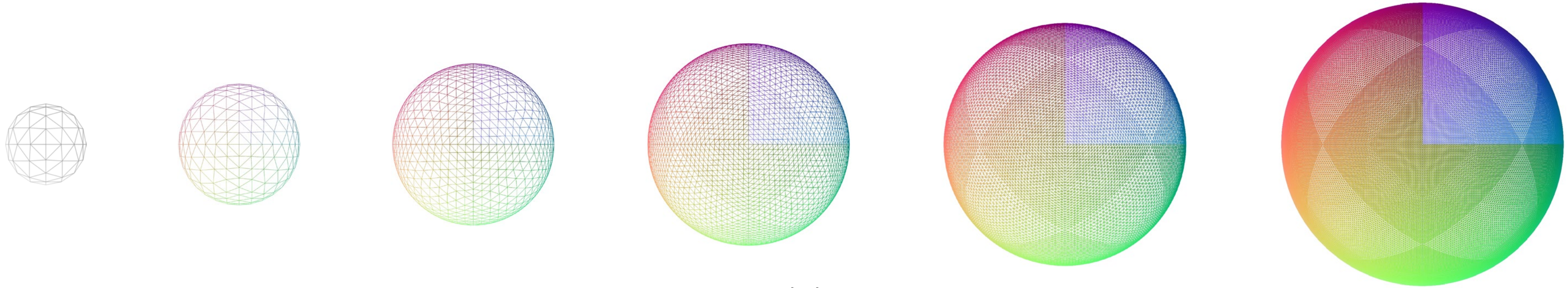
- Formulation (signal constancy):

$$E(\vec{v}) = \int_{\mathcal{S}^2} \left( f_s(p) - f_t\left(\exp_p \vec{v}(p)\right)\right)^2 dp$$

- Regularization (small motion constraint):

$$\epsilon \int_{\mathcal{S}^2} \|\vec{v}(p)\|^2 dp$$

- Coarse-to-fine hierarchy (sub-division):



# Overview

- Optical flow on the image plane ($\mathcal{I} \subset \mathbb{R}^2$)
- Optical flow on the spherical image ($S^2$)
- Optical flow on a 2D surface ($\mathcal{M}$)

# Optical flow – surfaces

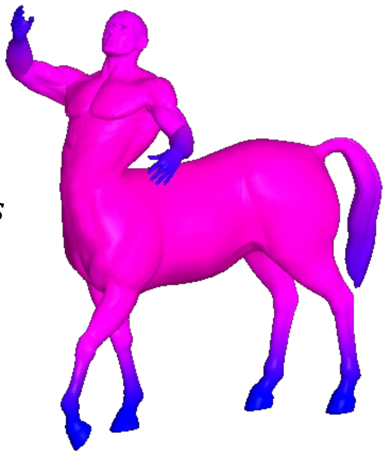
- Given two functions on a surface $\mathcal{M}$

$$f_s, f_t : \mathcal{M} \to \mathbb{R}$$

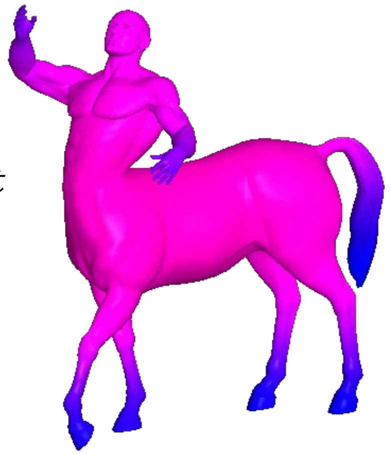
- An optical flow is a vector field $\vec{v} \in \Gamma(\mathrm{T}\mathcal{M})$ such that

$$f_s(p) = f_t(\exp_p \vec{v}(p))$$

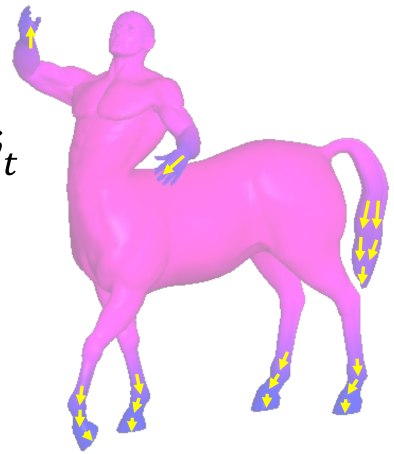
$f_s$



$f_t$



$\vec{v}_t$



# Optical flow – formulation

$$E(\vec{v}) = \int_{\mathcal{M}} \left(f_s(p) - f_t\left(\exp_p \vec{v}(p)\right)\right)^2 dp$$

- Using 1st order Taylor expansion:

$$f\left(\exp_p \vec{v}(p)\right) \approx f(p) + \langle \vec{v}(p), \nabla f(p) \rangle$$

where $\langle \cdot,\cdot \rangle$ is an inner product: $\mathrm{T}_p\mathcal{M} \times \mathrm{T}_p\mathcal{M} \to \mathbb{R}$

- Denoting $\delta(p) = f_s(p) - f_t(p)$, the energy becomes

$$E(\vec{v}) \approx \int_{\mathcal{M}} (\delta - \langle \vec{v}, \nabla f_t \rangle)^2 \, dp$$

# Optical flow – surfaces

Challenge:

Surfaces do not have a multi-resolutional structure we can leverage for a hierarchical solve.

Observation:

Using the hierarchical structure, we were implicitly enforcing smoothness on the representation of the **signal** and **vector field**.

Key Idea:

We can enforce the smoothness explicitly (without a multiresolution structure)

# Optical flow – coarse-to-fine refinement

For a given hierarchy level $l$:

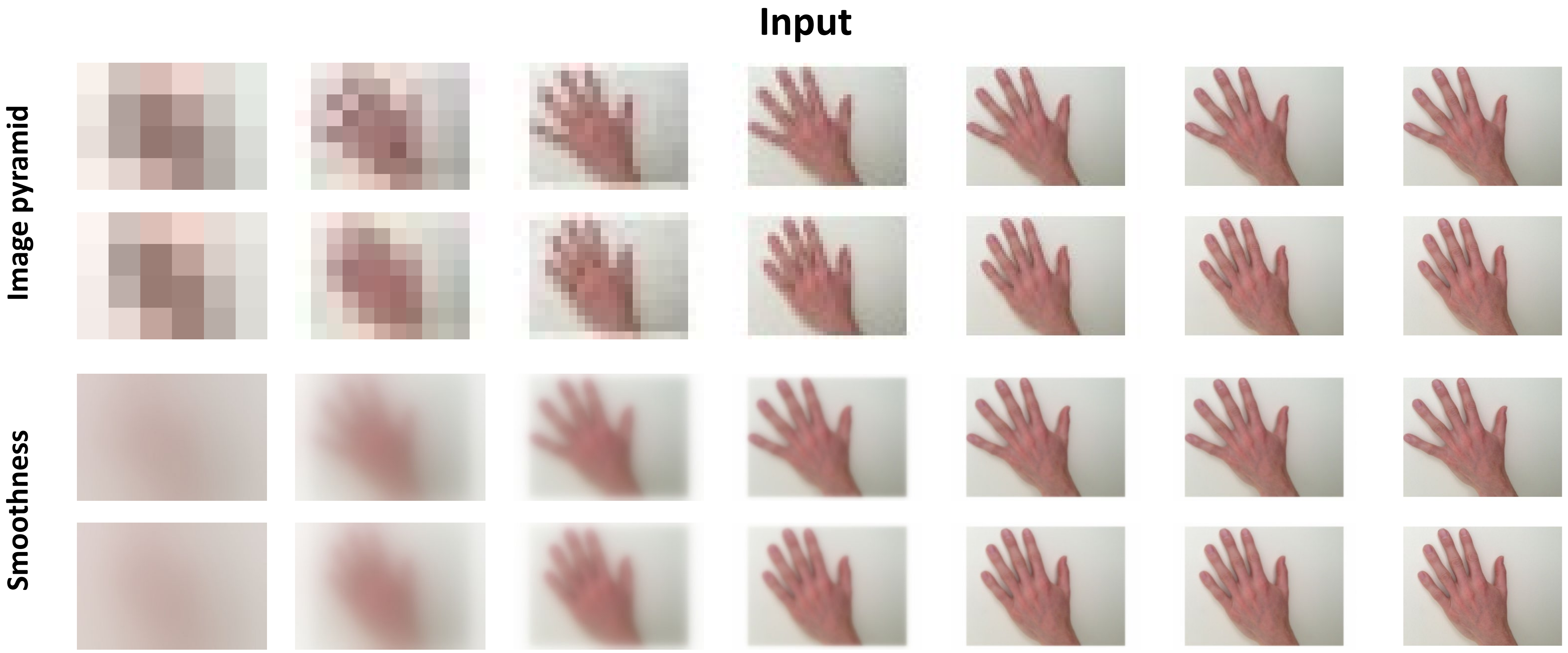
- Smooth the input signals $f_s, f_t$ with smoothness weights $\alpha^{(l)}$:

$$E\left(f^{(l)}\right) = \int_{\mathcal{M}} \left(f^{(l)} - f\right)^2 dp + \alpha^{(l)} \int_{\mathcal{M}} \left\|\nabla f^{(l)}\right\|^2 dp$$

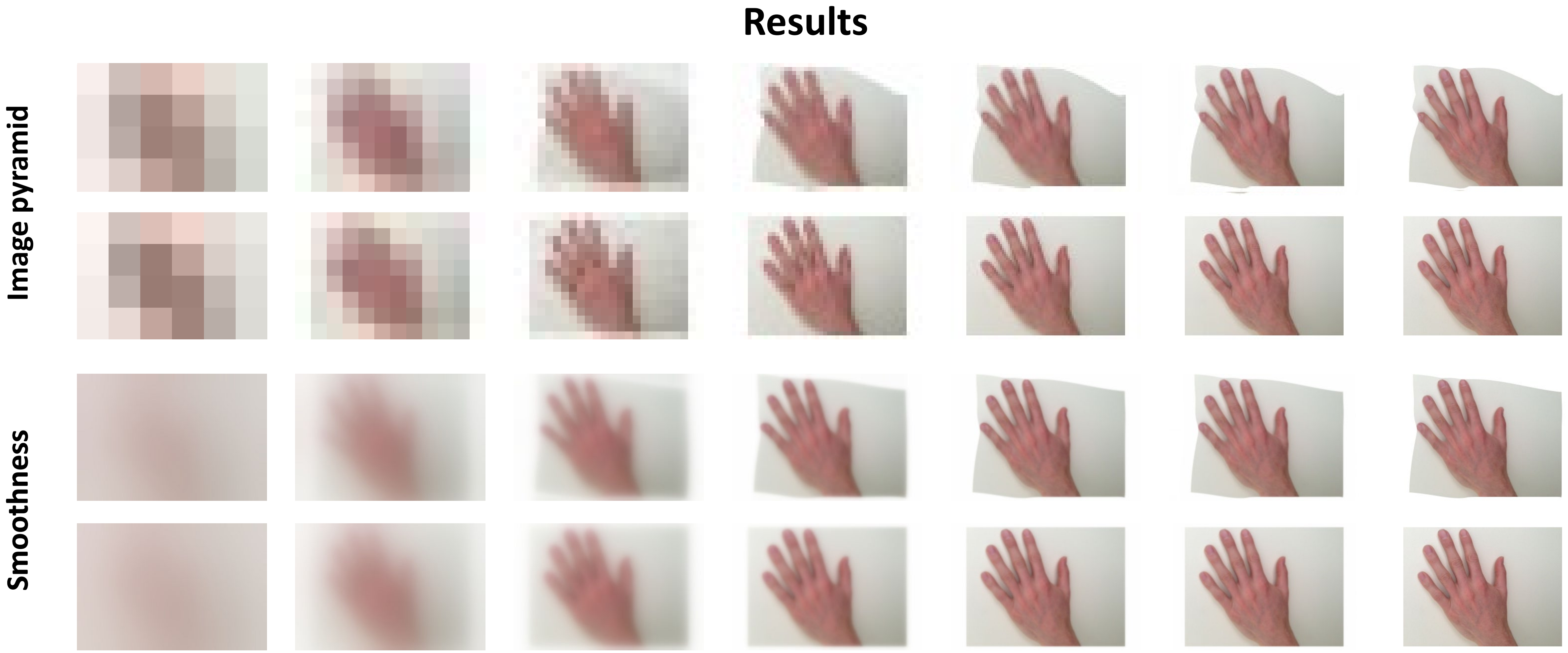
- Introduce a smoothness energy for the vector field:

$$E\left(\vec{v}^{(l)}\right) = \int_{\mathcal{M}} \left(\delta^{(l)} - \left\langle \vec{v}^{(l)}, \nabla f_t^{(l)} \right\rangle\right)^2 dp + \alpha^{(l)} \int_{\mathcal{M}} \left\|\nabla \vec{v}^{(l)}\right\|_F^2 dp$$

# Comparison – image pyramid vs smoothness



# Comparison – image pyramid vs smoothness

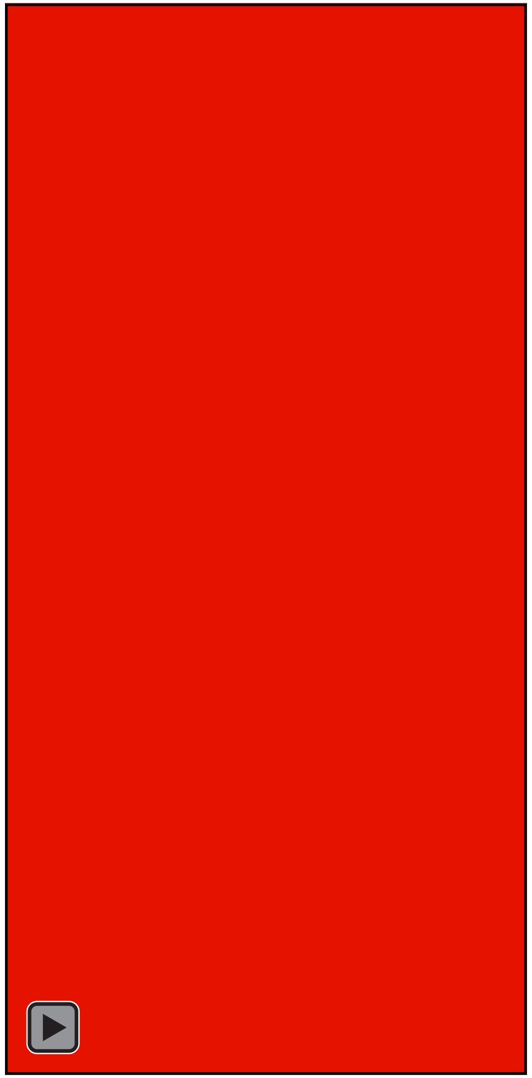


# Comparison – image pyramid vs smoothness

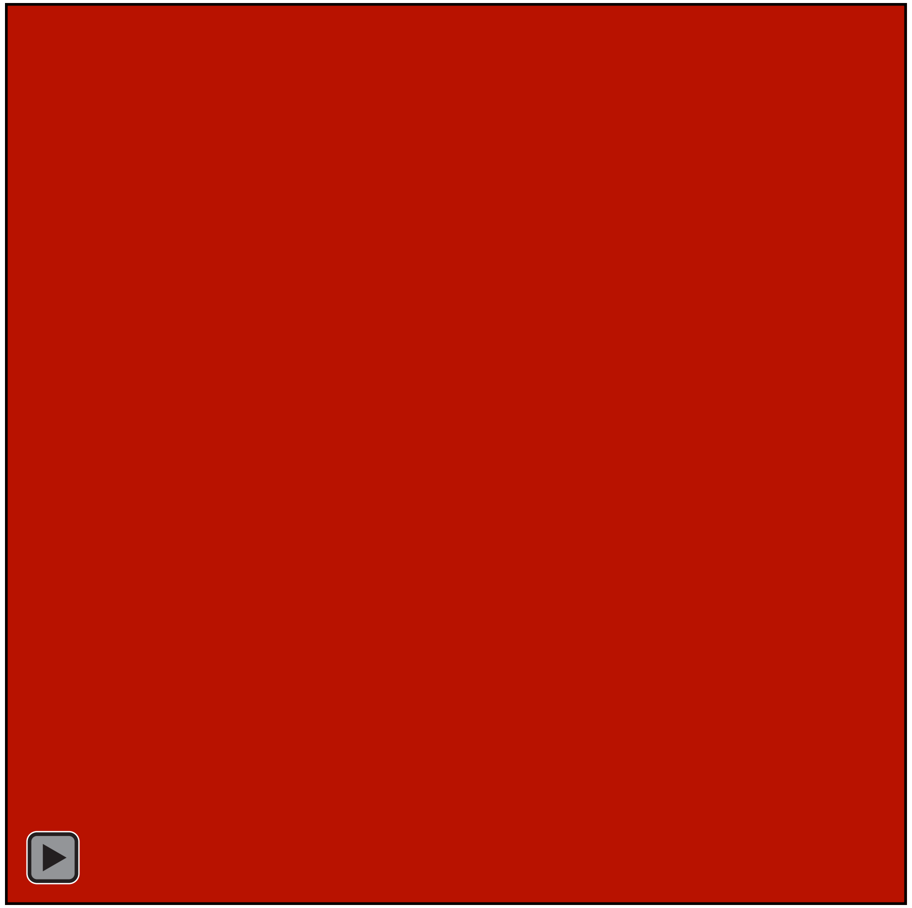
**Image pyramid**

**Smoothness**

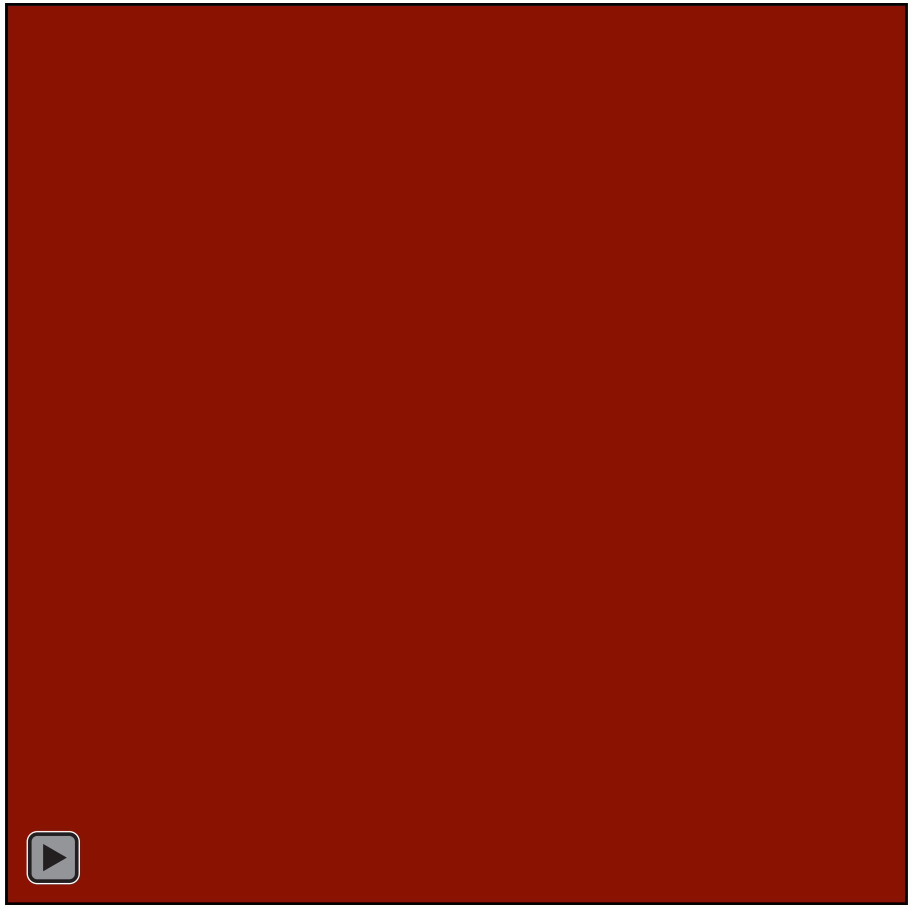
# Application – texture interpolation



**Source**

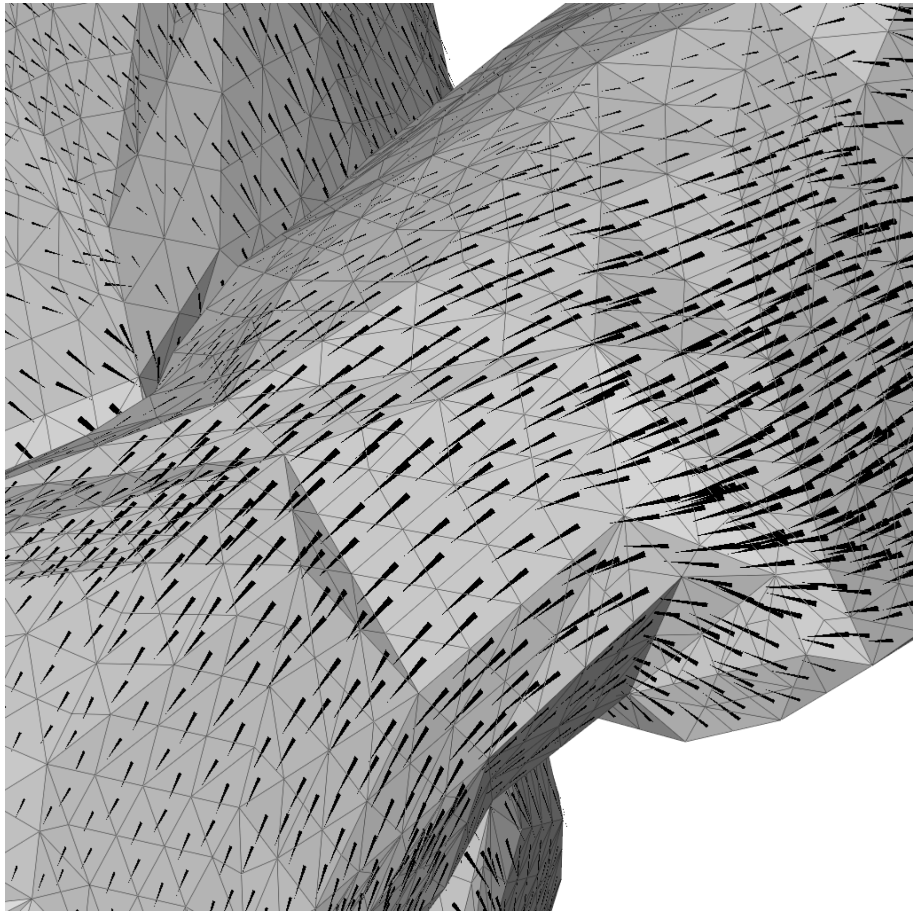


**Target**

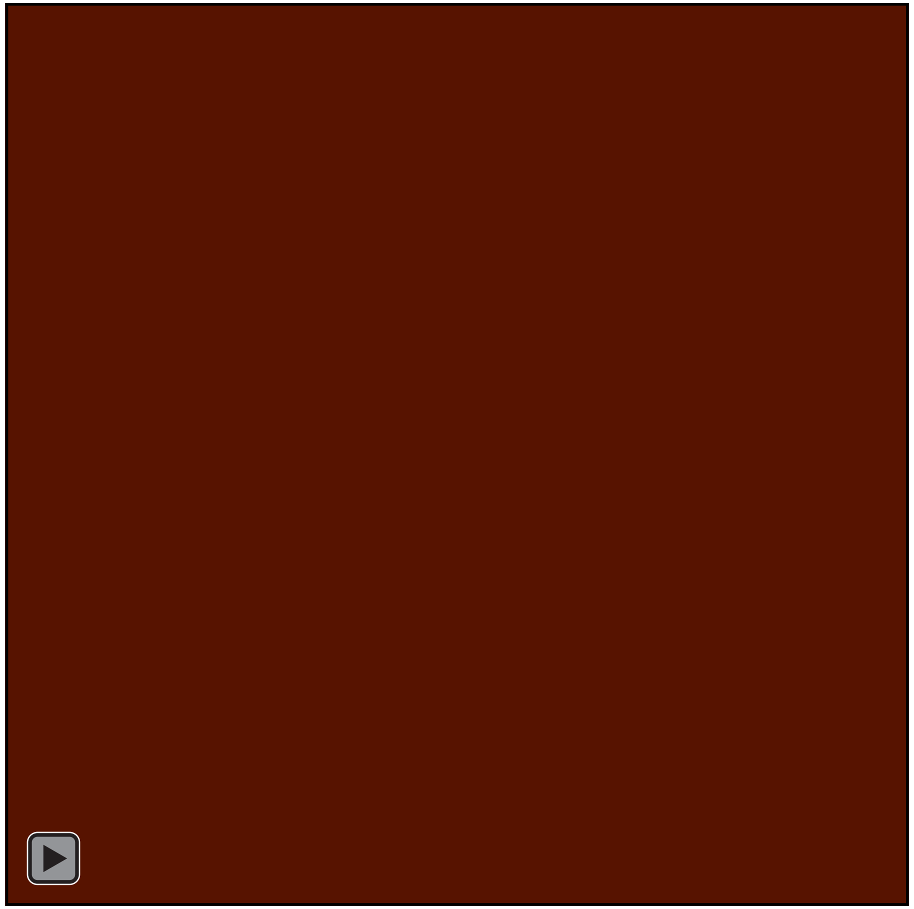


# Application – texture interpolation

**Optical flow vector field**



**Source to target advection**



# Optical flow – surfaces

- Formulation (signal constancy):

$$E(\vec{v}) = \int_{\mathcal{M}} \left(f_s(p) - f_t\left(\exp_p \vec{v}(p)\right)\right)^2 dp$$

- Regularization (small motion constraint):

$$\epsilon \int_{\mathcal{M}} \|\vec{v}(p)\|^2 dp$$

- Coarse-to-fine hierarchy (smoothed signals + smoothness constraint):

$$\alpha^{(l)} \int_{\mathcal{M}} \left\|\nabla \vec{v}^{(l)}\right\|_F^2 dp$$

# Thank you!