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Shape Correspondence on the Unit Sphere

Sing Chun Lee[1], Misha Kazhdan[1]

Preparing for Submission to Symposium on Geometry Processing 2019

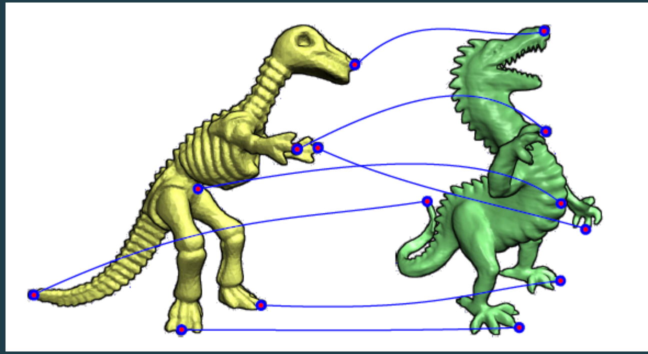
[1]Johns Hopkins University

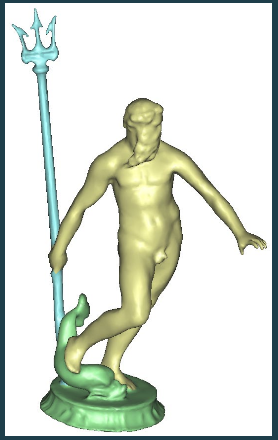
Outline

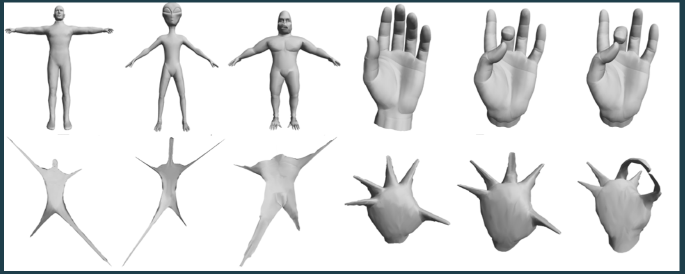
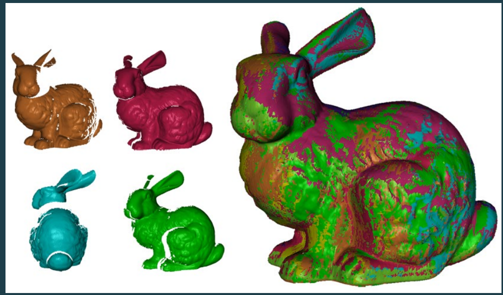
- Problem Description
- Related Works
- Review of Mathematical Tools
- Problem Formulation
- Our Approach
- Current Results and Demo
- Q&A and Feedback

Problem Description

- Shape Correspondence
  - Given two shapes $S_1, S_2 \subset \mathbb{R}^3$, we want to find meaningful map $\Xi : S_1 \rightarrow S_2$ such that $\{(s, \Xi(s) | s \in S_1, \Xi(s) \in S_2\}$ give us a meaningful correspondence between them





*Pictures borrowed from: O. Kaick, D. Cohen-Or et al., "A Survey on Shape Correspondence", in *CGF 2011*

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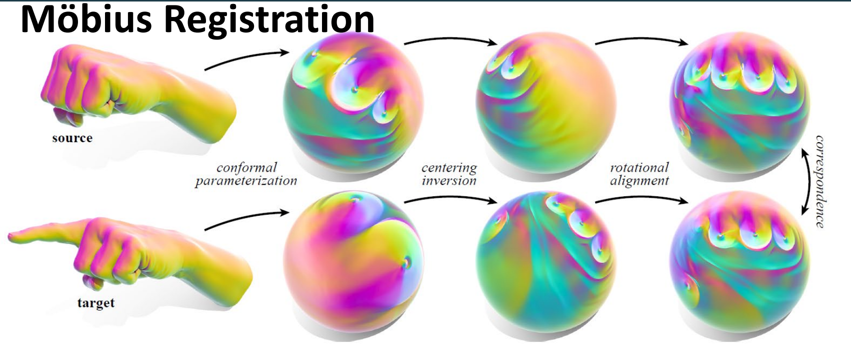
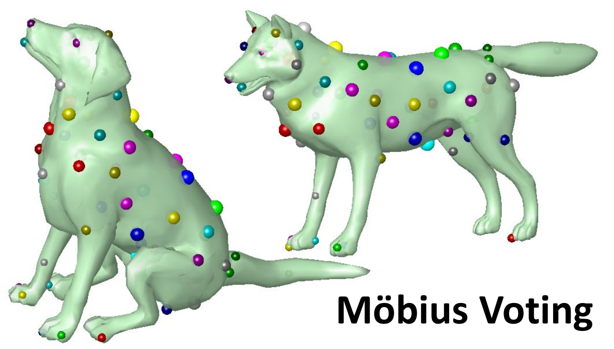
Method Taxonomy

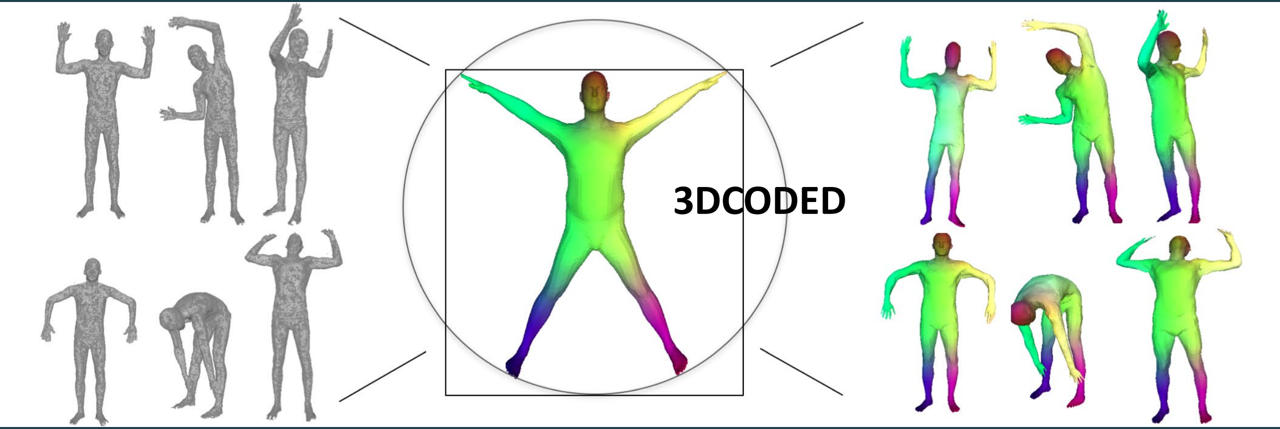
- Output:
  - ***Full*** vs. Partial
  - ***Dense*** vs. Sparse
- Objective function:
  - Rigid vs. ***Deformation***
- Approach:
  - ***Fully-automatic*** vs. Semi-automatic
  - Global vs. ***Local***
  - ***Pairwise*** vs. Groupwise

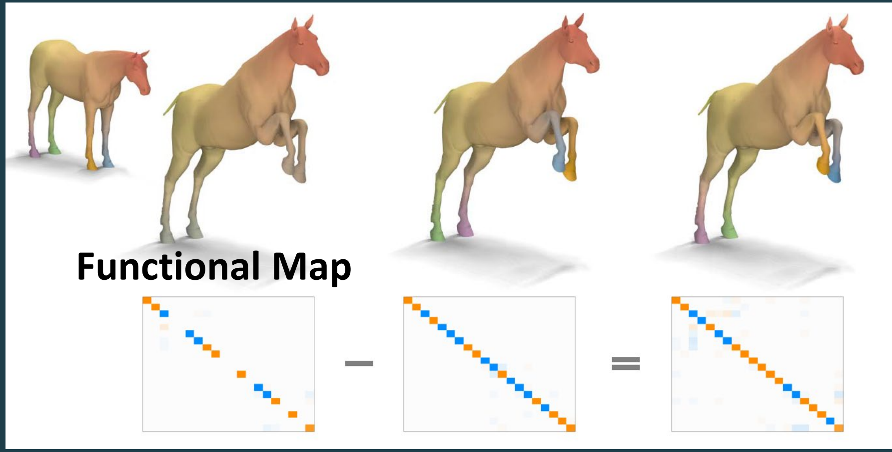
Related Works

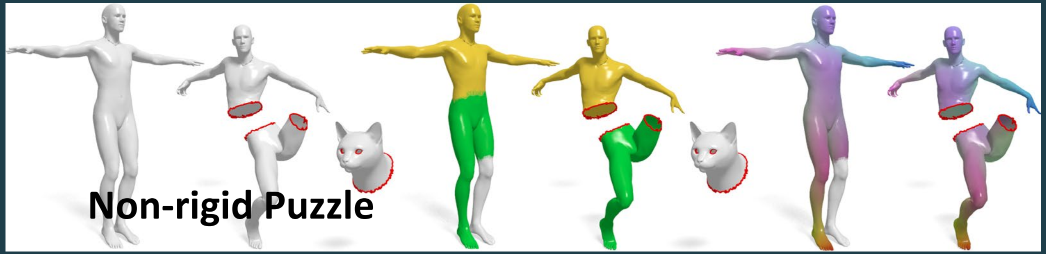
- Möbius Voting
- Möbius Registration
- Functional Map
- Non-rigid Puzzle
- 3DCODED





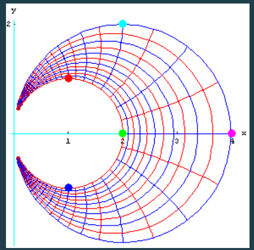


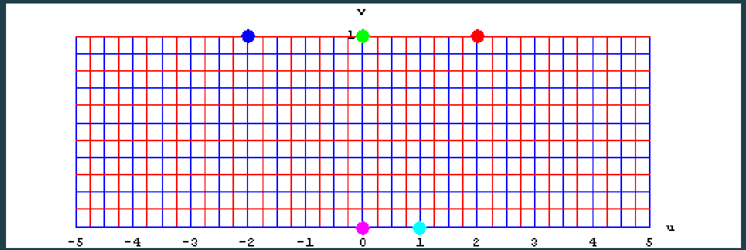


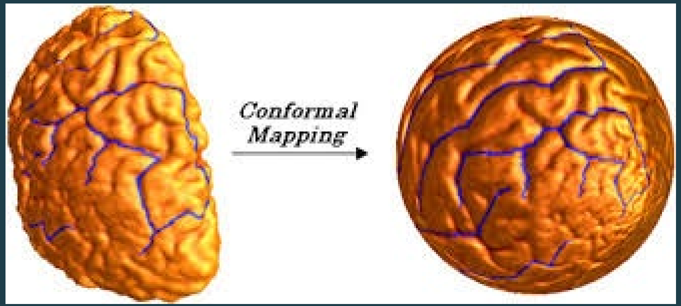


*Pictures borrowed from the captioned publications

Möbius Transformation





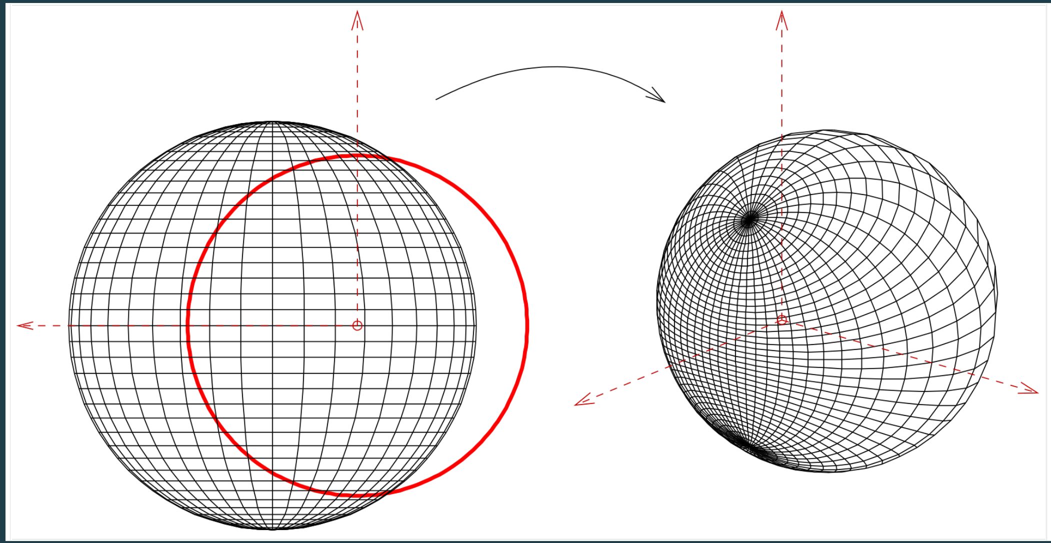


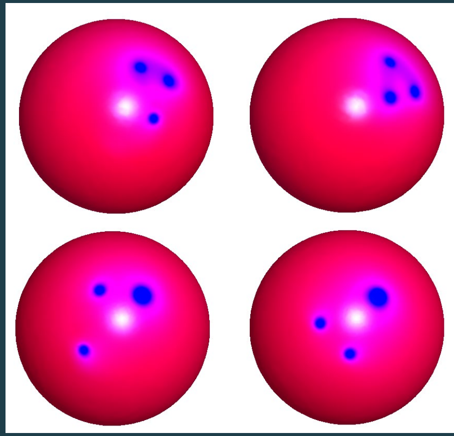
- Aka. Homographies
- General form in $\mathbb{C}$ $(ad - bc \neq 0)$:
  - $f(z) = \frac{az+b}{cz+d}$
- Matrix form $\mathbb{C}^{2\times2}$ $(\det \neq 0)$:
  - $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- It’s conformal (angle-preserved): map line to a line or circle, and map circle to a line or circle
- Area ratio before/after transformation: conformal factors
- A tool to map genus zero shape (S) to the unit sphere $(\mathbb{S}^2)$
  - Denote by $\Phi: S \rightarrow \mathbb{S}^2$

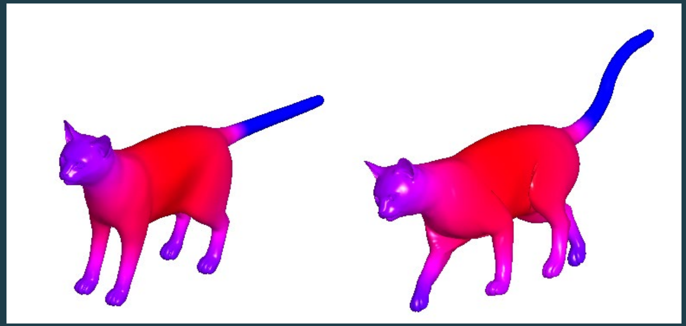
*Pictures borrowed from “Landmark constrained genus zero surface conformal mapping and its application to brain mapping research, IMACS 2006

Möbius Registration via Spherical Inversion

- $\eta_c : \mathbb{S}^2 \to \mathbb{S}^2$
  - $\eta_c \coloneqq (1 - |c|^2)\frac{p+c}{|p+c|^2} + c$
- Unique Möbius transformation (up to Rotation) by finding η_c such that the average of the conformal factors is zero
- i.e. center of mass is at the origin
- Möbius registration = aligning conformal factors for rotation







Generalized Optical Flow on Surface

- A generalized version of optical flow on surface is defined as follows: given two signals $f_i: S_i \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ on two surface meshes S_i with correspondence, we can estimate an optical flow $\vec{v}$ by minimizing:

$$E(\vec{v}) = \int_{\mathbb{R}^3} \|\langle \nabla f_1(p), \vec{v}(p)\rangle - \delta(p)\|^2 + \|\langle \nabla f_2(p), \vec{v}(p)\rangle - \delta(p)\|^2 + \epsilon \|\nabla \vec{v}(p)\|^2 dp$$

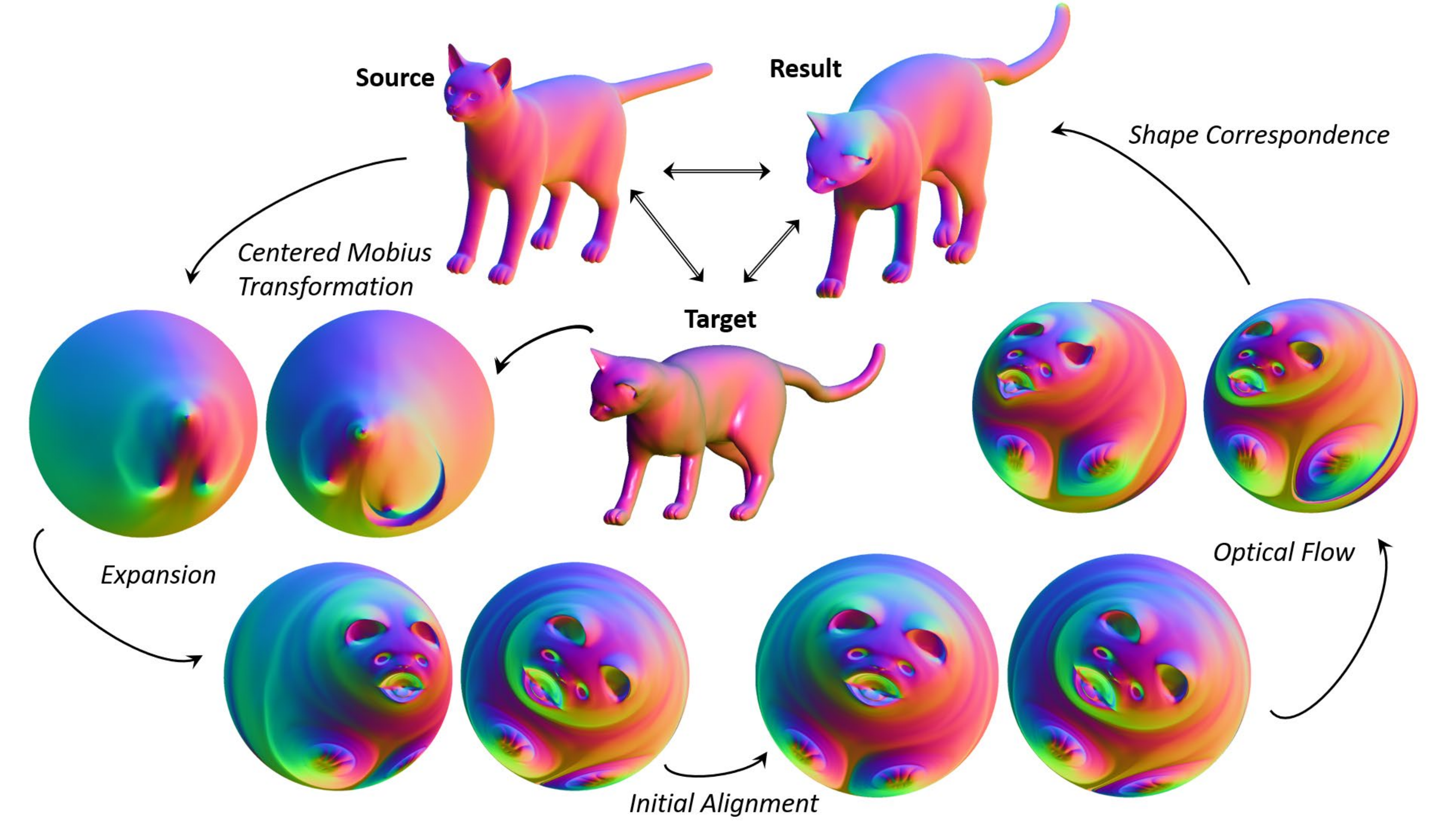
where $\langle \nabla f_i(p), \vec{v}(p)\rangle$ is the advection of the signal between two meshes, $\delta(p) = (f_1(p) - f_2(p)$ is the difference between two signals, and $\|\nabla \vec{v}(p)\|$ is the smoothness term to regularize the vector field $\vec{v}$

Problem Formulation

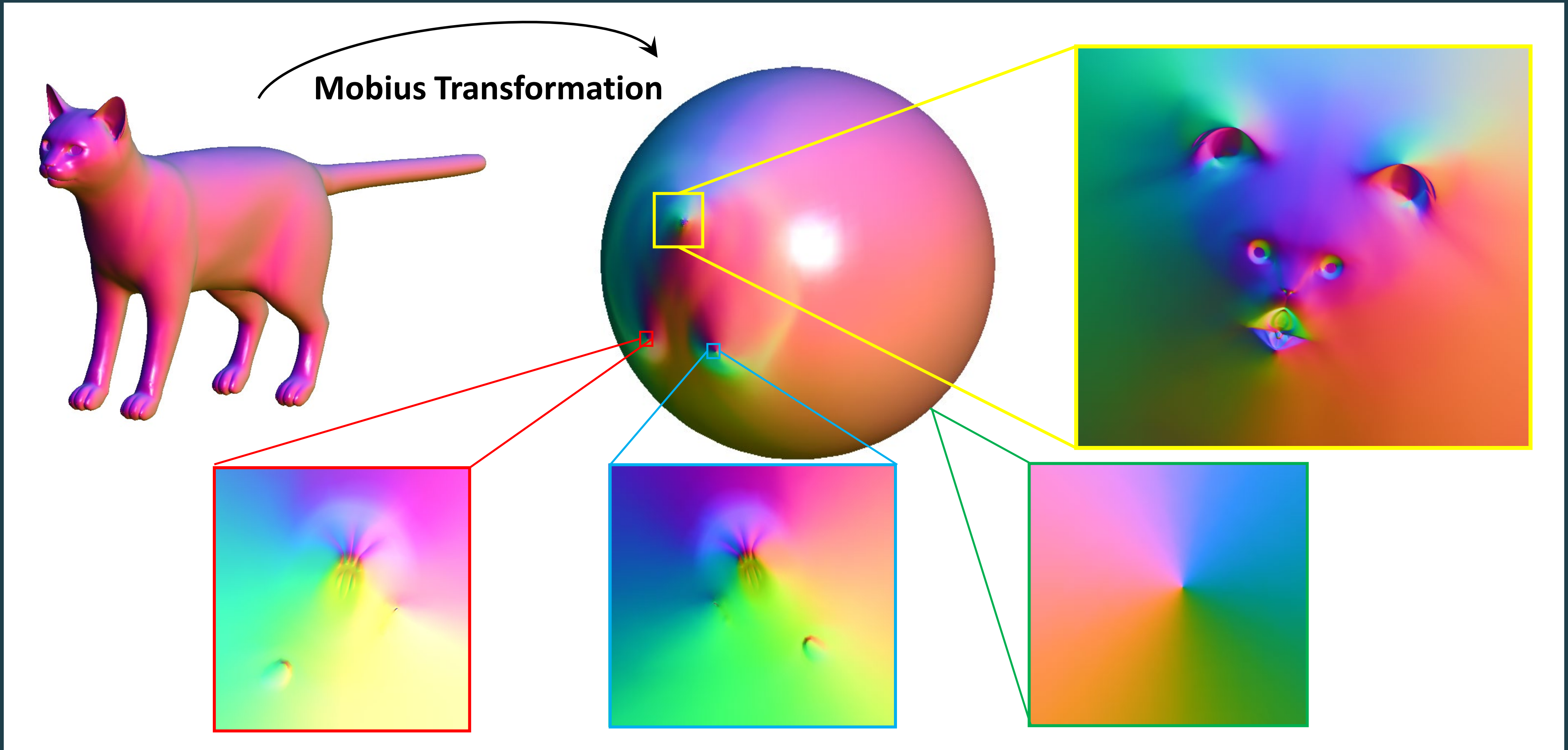
- Given two shapes $S_1, S_2 \subset \mathbb{R}^3$, corresponding signals $f_i : S_i \to \mathbb{R}^d$
- Compute and sample f_i of S_i on $\mathbb{S}^2$ via unique centered Möbius transformation, namely $\Phi_i : S_i \to \mathbb{S}^2$,
  i.e. pullback to $\mathbb{S}^2$ - $g_i \coloneqq f_i \circ \Phi_i^{-1} : \mathbb{S}^2 \to \mathbb{R}^d$
- Perform initial alignment (a rotation) that minimize
  - $E(R) = \min_R \int_{\mathbb{S}^2} \| g_1 \circ R(p) - g_2(p) \|^2 \, dp$
- Estimate optical flow that minimize
  - $E(\vec{v}) = \int_{\mathbb{S}^2} \| \langle \nabla g_1 \circ R(s), \vec{v}(s) \rangle - \delta(s) \|^2 + \| \langle \nabla g_2(s), \vec{v}(s) \rangle - \delta(s) \|^2 + \epsilon \| \nabla \vec{v}(s) \|^2$

where $\delta(s) = g_1 \circ R(s) - g_2(s)$

Overview


Source
Result
Shape Correspondence
Centered Mobius
Transformation
Target
Expansion
Optical Flow
Initial Alignment

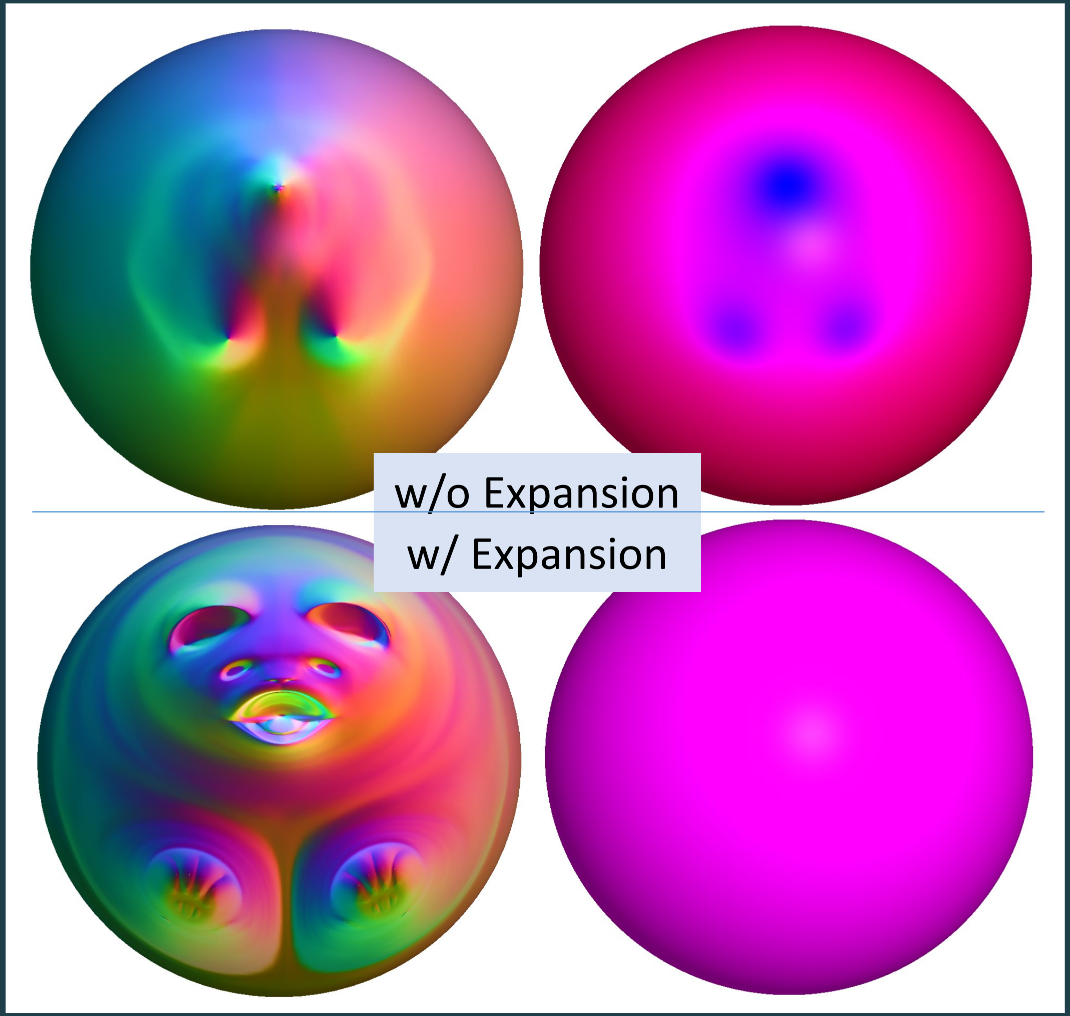
Step 1 – Möbius Transformation



Step 2 – Expansion

- Problem: dense region is too packed ➔ hard to discretize
- Idea: expand the dense region out and shrink the sparse region
- How? Using Poisson Equation
- Given a conformal factor $\phi(x)$, we would like to uniformized the values
- Goal: find a vector field $\vec{v}$ such that $\phi(x + t\vec{v}(x))$ is uniform
- Solve: $\Delta\phi(x + t\vec{v}(x)) = \frac{d}{dt}\phi(x + t\vec{v}(x))$ at $t = 0$
- Solution: $\vec{v}(x) = -\frac{2\Delta\phi}{\phi} = -2\Delta\log\phi$

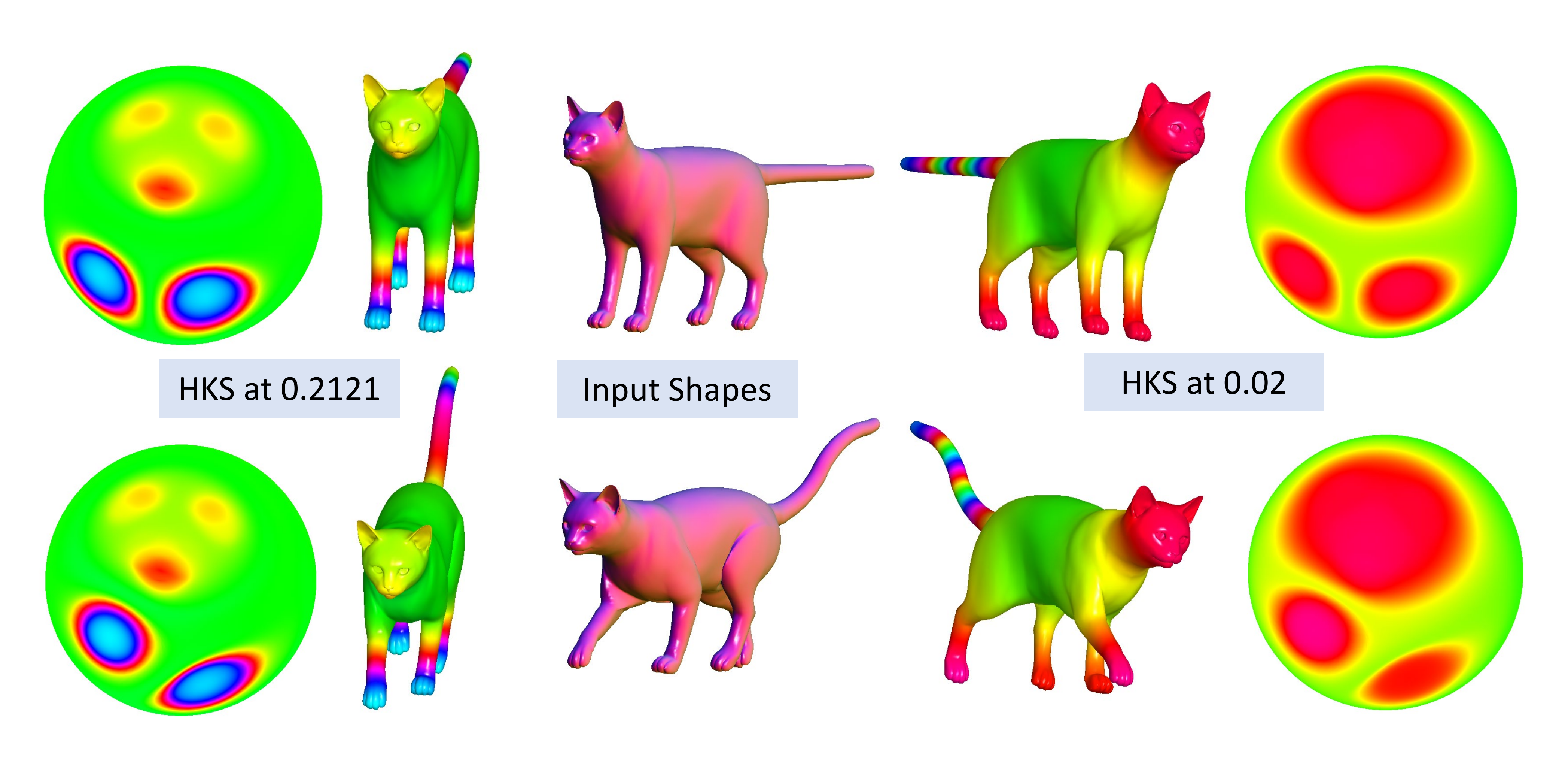
Expansion Result



Step 3 – Corresponding Signals

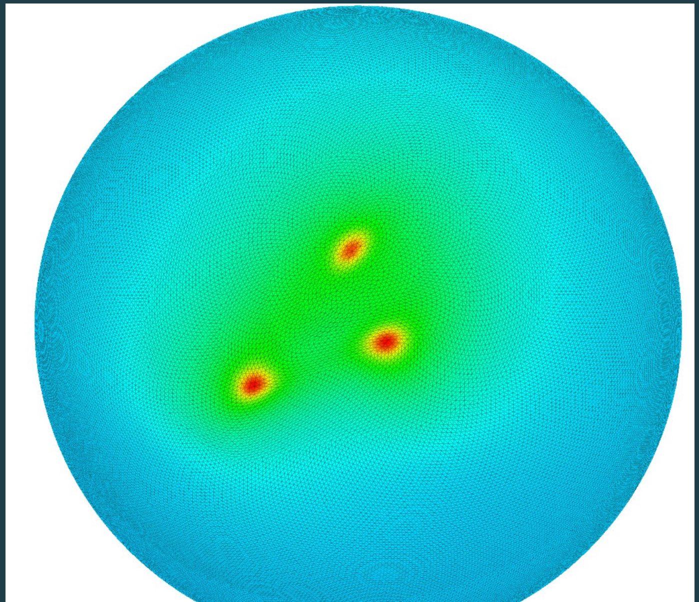
- Isometric invariant signatures
  - Heat Kernel Signature (HKS)
- Eigenvalues and Eigenvectors of Generalized Eigenvalue Decomposition of the Shape Operator
  - Solving $Sx = \lambda Mx$ where S and M are the stiffness and mass matrices
  - Algorithm: Arnoldi iteration
- Let's α_i be the eigenvector associated with eigenvalue λ_i
- HKS is defined by:
  - $h_t(x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \alpha_i(x) \alpha_i(y)$

Step 3 – Computed Signals

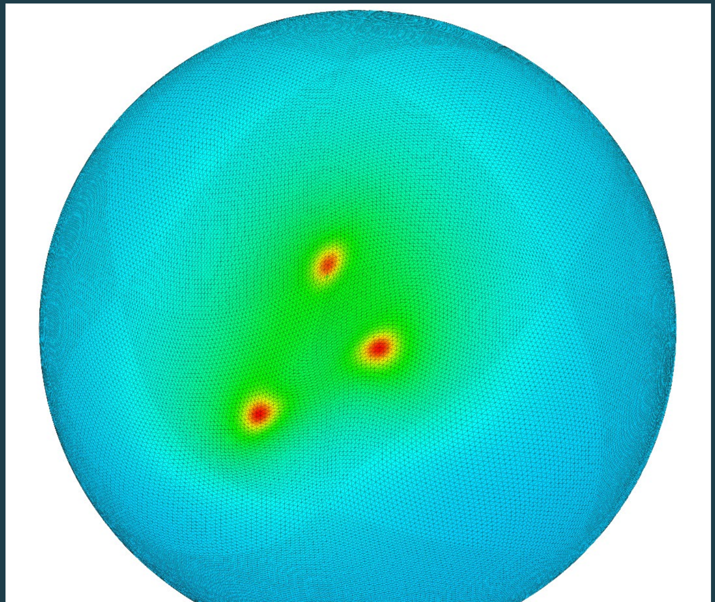


Step 4 – Sampling on the Sphere

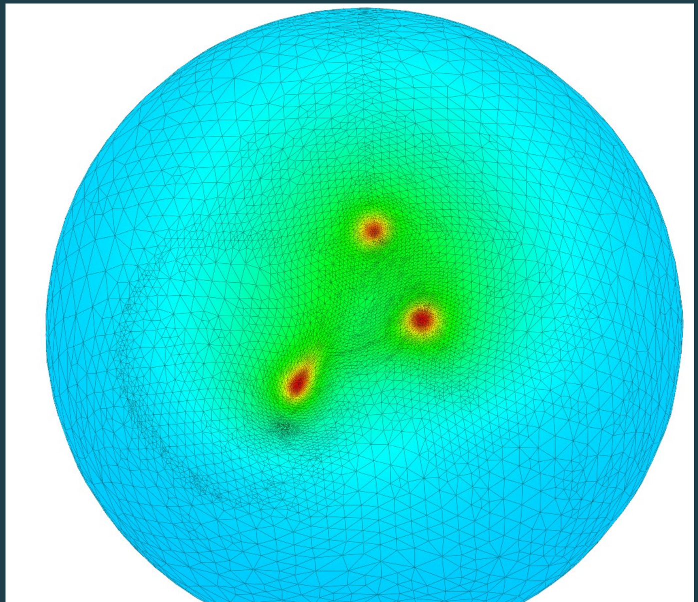
- Fibonacci Sampling
- Octahedron Sampling
- Adaptive Octahedron Sampling



Fibonacci



Octahedron

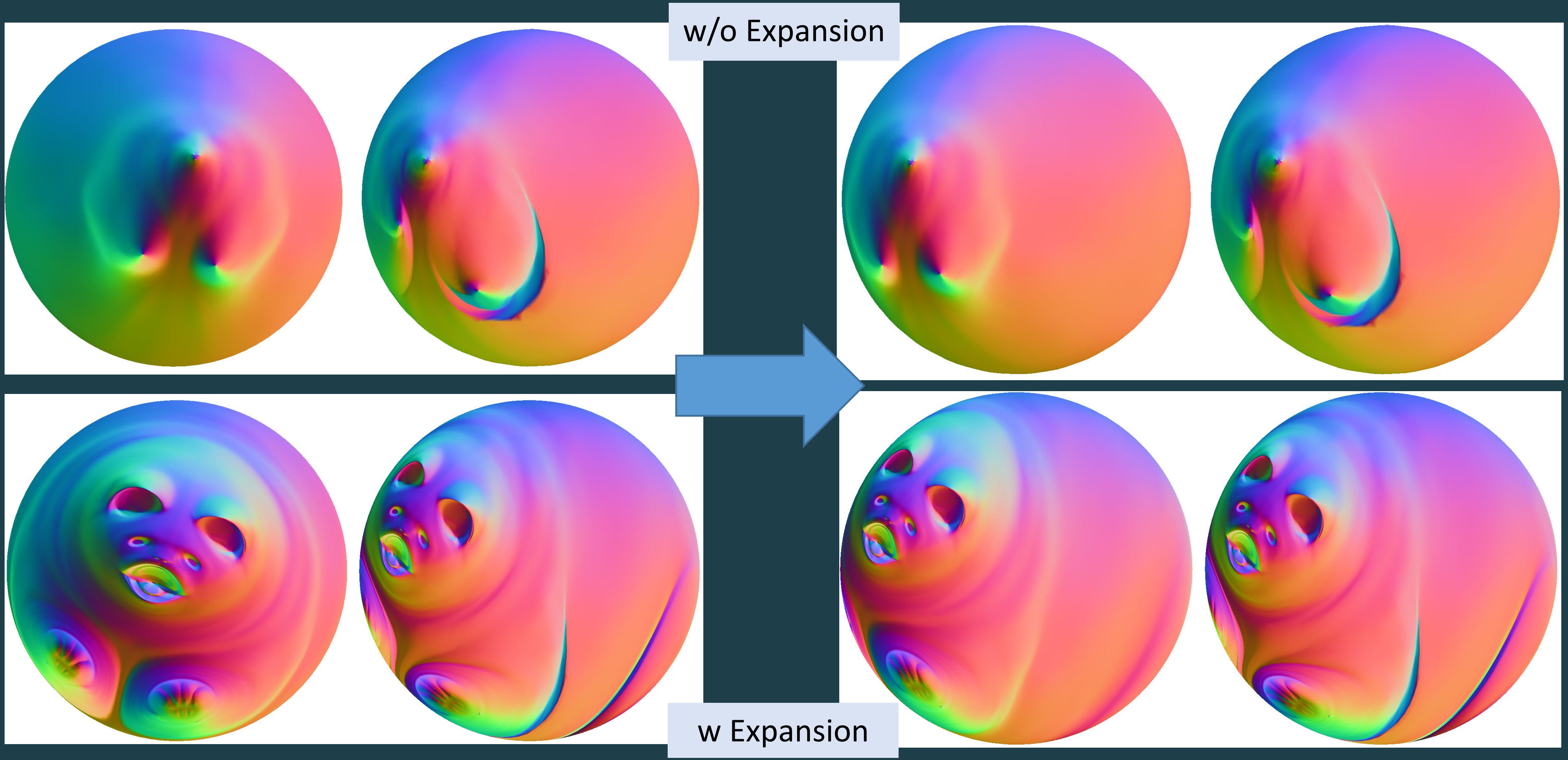


Adaptive Octahedron

Step 5 – Initial Alignment

- Harmonic Transform for efficient maximizing of correlation
  - $E(\rho_R) = \min_{\rho_R} \|g_1 - \rho_R(g_2)\|^2$
  - $E(\rho_R) = \min_{\rho_R} \|g_1\|^2 + \|g_2\|^2 - 2\langle g_1, \rho_R(g_2)\rangle$
  - $E(\rho_R) = \max_{\rho_R} \langle g_1, \rho_R(g_2)\rangle$
- It is equivalent to correlation, which can done efficiently using Harmonic Transform for the sphere

Alignment Result



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Initial Correspondence

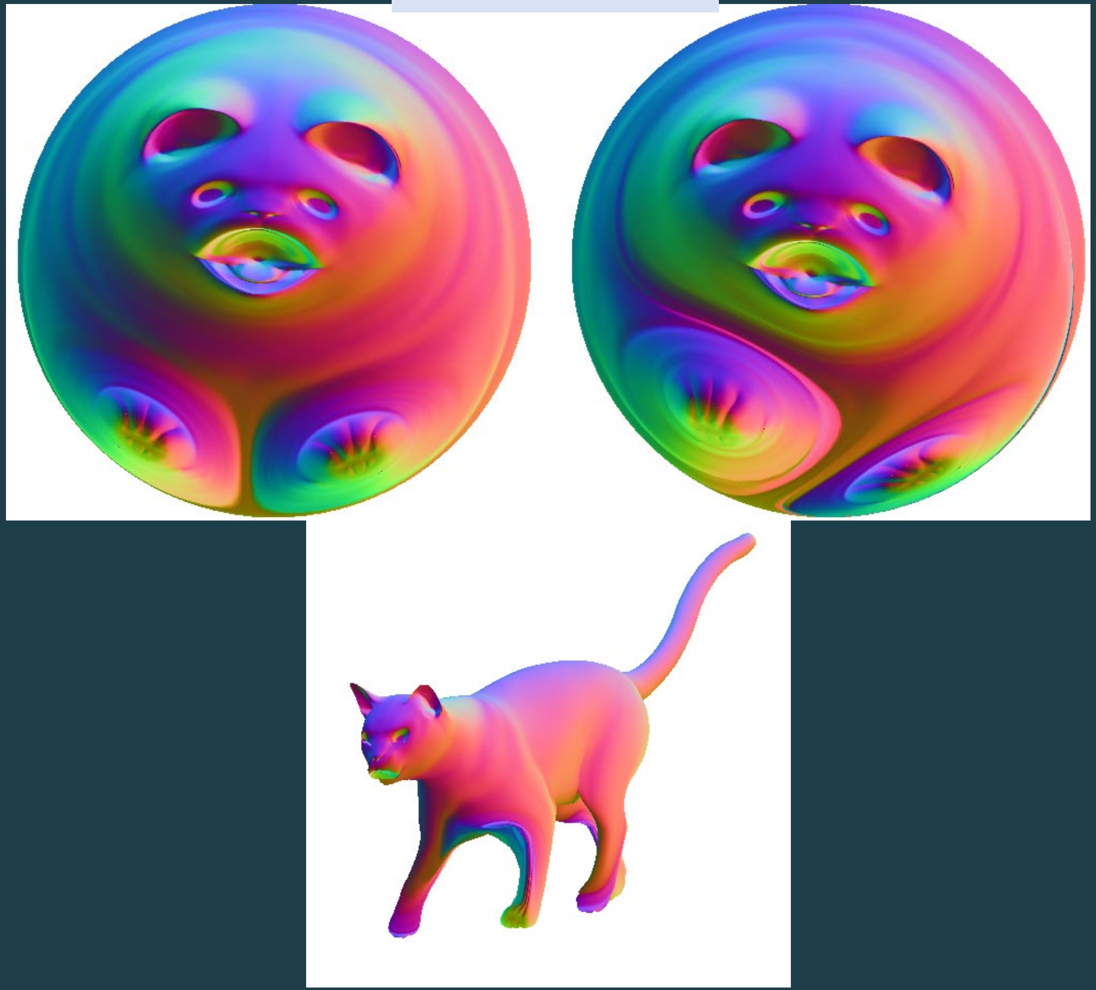
w/o Expansion | w Expansion

Step 5 – Surface Optical Flow

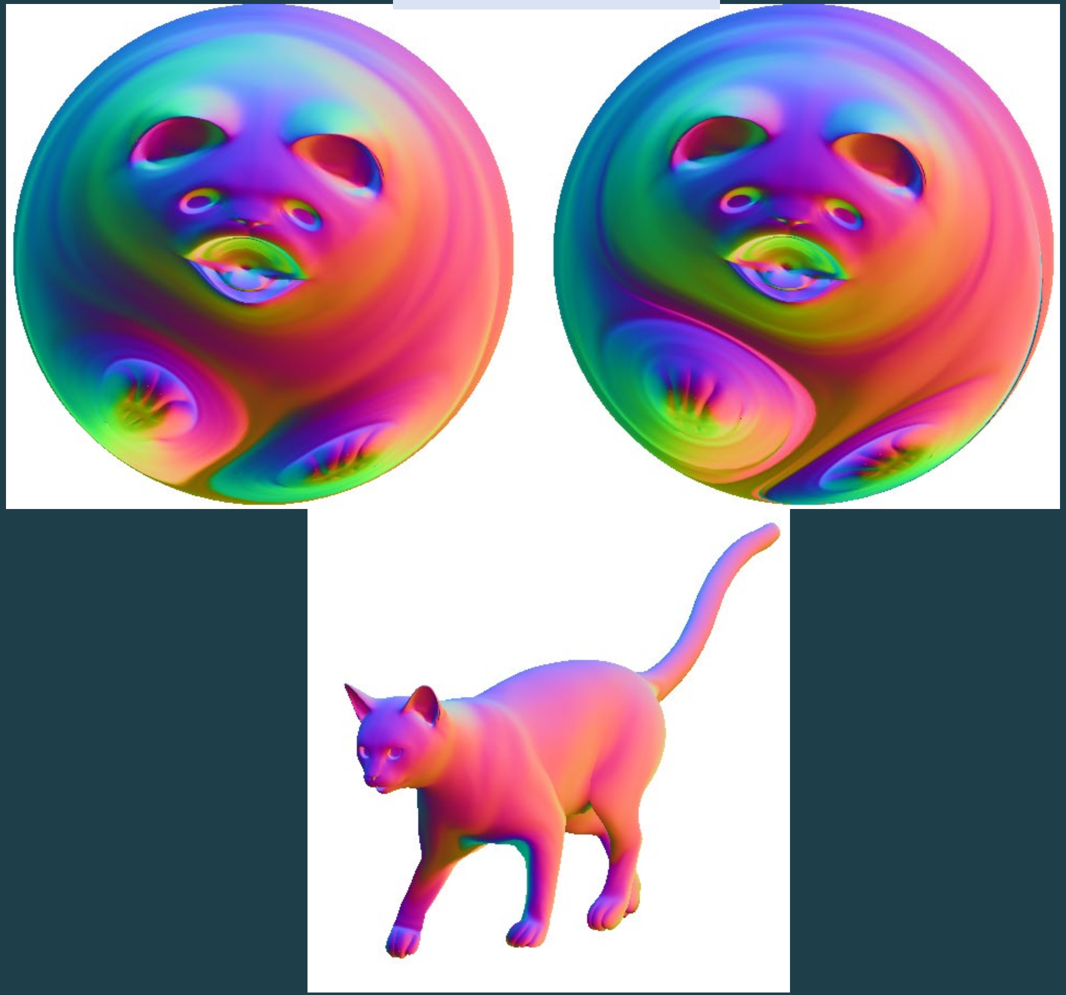
- Recall:
  - $E(\vec{v}) = \int_{\mathbb{S}^2} \|\langle \nabla g_1 \circ R(s), \vec{v}(s)\rangle - \delta(s)\|^2 + \|\langle \nabla g_2(s), \vec{v}(s)\rangle - \delta(s)\|^2 + \epsilon \|\nabla \vec{v}(s)\|^2$
- Signals (g_i) are HKSs at different time steps
  - Isometric invariant
- But: shapes also undergo non-isometric deformation
- Additional constraint:
  - $E(\vec{v}) = \int_{\mathbb{S}^2} \sum_{i=0,1} \|\langle \nabla g_i(s), \vec{v}(s)\rangle - \delta(s)\|^2 + \epsilon \|\nabla \vec{v}(s)\|^2 + \gamma \|\nabla \cdot \vec{v}(s)\|^2$
  - (With abuse of notation, we write $\nabla g_1(s) = \nabla g_1 \circ R(s)$)
  - Meaning: Make the vector field preserve the triangle area (as div = 0)

Step 5 – Surface Optical Flow

Before



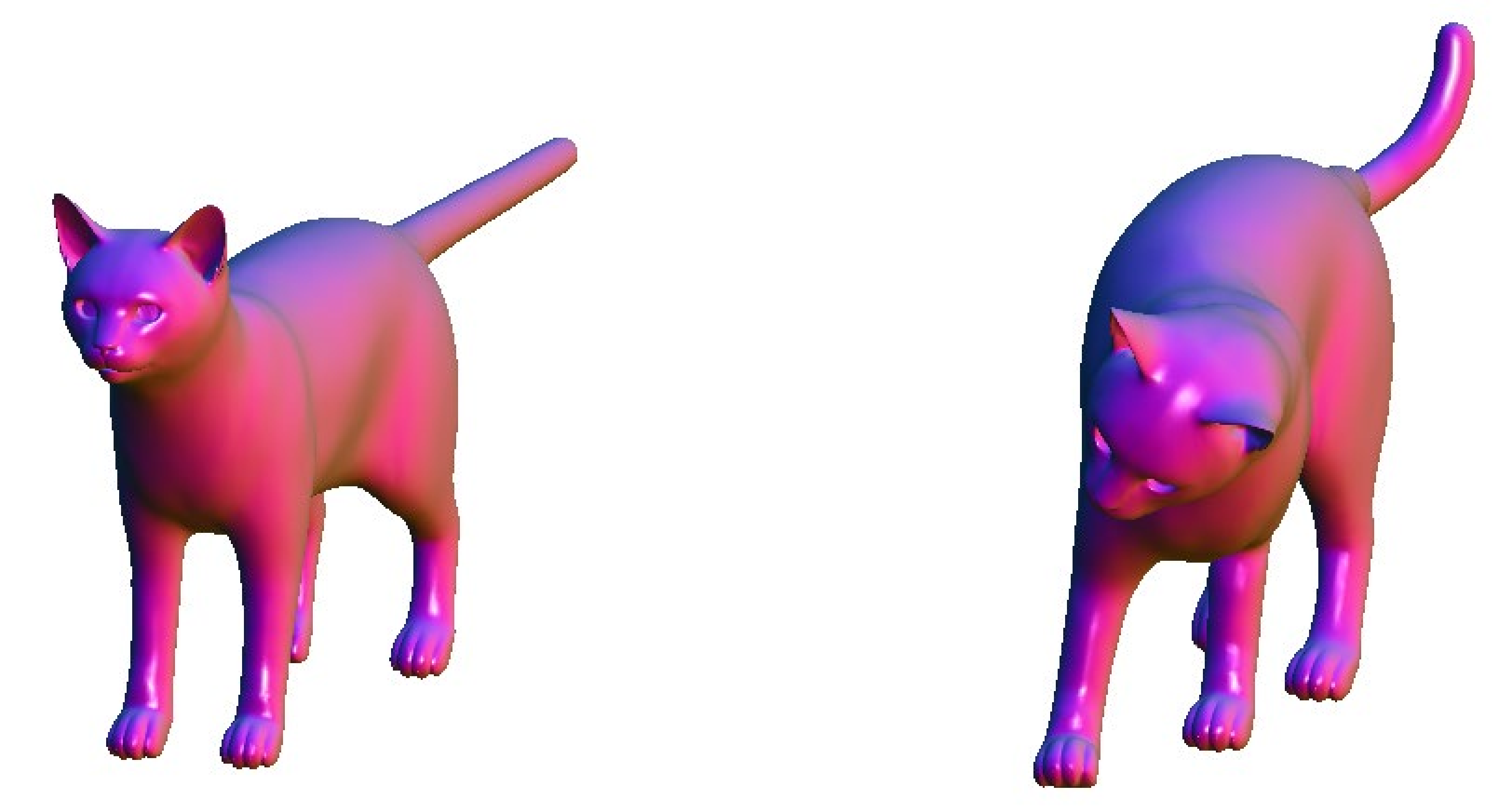
After



Step 5 – Sphere Optical Flow

Effect of Divergence Constraint

- Shapes having more non-isometric deformation



Without Constraint

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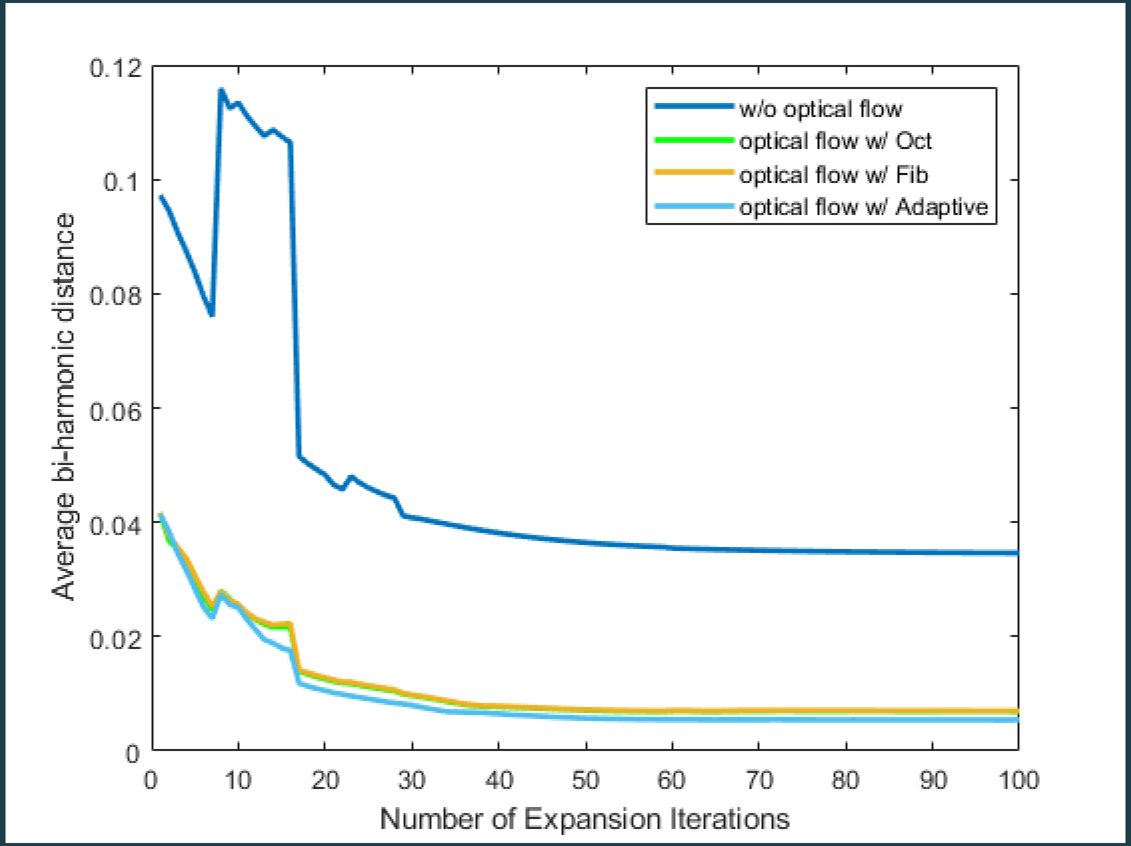
With Constraint

Comparison


0.12
0.1
0.08
0.06
0.04
0.02
0
Average bi-harmonic distance
0
10
20
30
40
50
60
70
80
90
100
Number of Expansion Iterations
w/o optical flow
optical flow w/ Oct
optical flow w/ Fib
optical flow w/ Adaptive

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Demo and Q&A

Feedback?