Hierarchical Gradient-Domain Vector Field Processing

Capital Graphics 2023

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Outline

- Gradient-domain scalar field processing
- Extending to vector fields
- Challenges
- Current solutions

Gradient-Domain Scalar Field Processing

- In gradient-domain processing, we solve for a scalar field ϕ on a triangle mesh $\mathcal{M}$ by minimizing

$$E(\phi) = \boxed{\|\phi - \psi\|^2} + \alpha\boxed{\|d\phi - \omega\|^2}$$

where

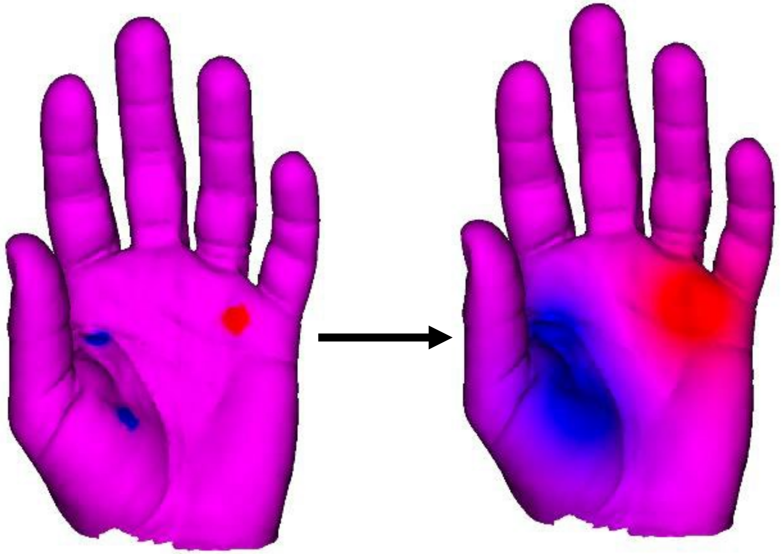
  - ψ: the target scalar field
  - ω: the target differential
  - α: the smoothness weight

[F. Prada, M. Kazhdan, M. Chuang, H. Hoppe. SIGGRAPH 2018]

Scalar Field Gradient-Domain Processing

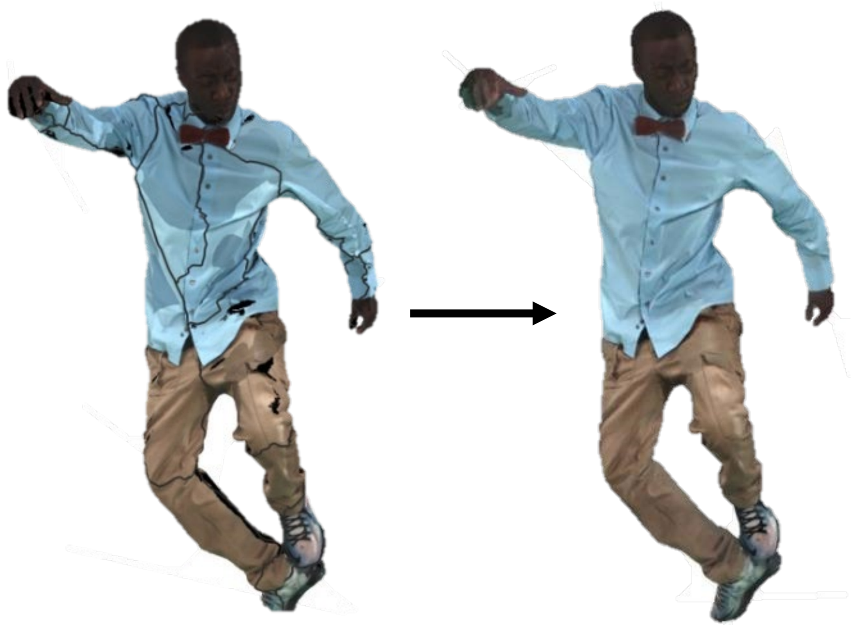
$$E(\phi) = \|\phi - \psi\|^2 + \alpha\|d\phi - \omega\|^2$$

Scalar field diffusion



Geodesic in heat

Stitching



[F. Prada, M. Kazhdan, M. Chuang, H. Hoppe. SIGGRAPH 2018]

Scalar Field Gradient-Domain Processing

- Discretizing using basis functions $\{\phi_v\}_{v\in V}$ gives:

$$\arg\min_{\phi} E(\phi) = \|\phi - \psi\|^2 + \alpha\|d\phi - \omega\|^2$$
$$\Downarrow$$
$$\arg\min_{\mathbf{x}} E(\mathbf{x}) = (\mathbf{x} - \mathbf{y})^\top \mathbf{M}(\mathbf{x} - \mathbf{y}) + \alpha(\mathbf{x}^\top \mathbf{S}\mathbf{x} - 2\mathbf{x}^\top \mathbf{z} + \mathbf{c})$$
$$\Downarrow$$
$$(\mathbf{M} + \alpha\mathbf{S})\mathbf{x} = \mathbf{M}\mathbf{y} + \alpha\mathbf{z}$$

Solving a
Linear System
$\mathbf{Ax} = \mathbf{b}$

where $\mathbf{x}, \mathbf{y}$ are coefficients vectors of ϕ, ψ w.r.t. $\{\phi_v\}_{v\in V}$, $\mathbf{z}, \mathbf{c}$ are some constant in terms of ω

Scalar Field Gradient-Domain Processing

$$(\mathbf{M} + \alpha\mathbf{S})\mathbf{x} = \mathbf{My} + \alpha\mathbf{z}$$

- $\mathbf{M}$: the mass matrix

$$\mathbf{M}_{v,v'} = \int_{\mathcal{M}} \phi_v \cdot \phi_{v'} \, dp$$

- $\boldsymbol{S}$: the stiffness matrix

$$\mathbf{S}_{v,v'} = \int_{\mathcal{M}} \langle d\phi_v, d\phi_{v'} \rangle \, dp$$

If we have scalar basis functions $\{\phi_v\}_{v \in V}$ on $\mathcal{M}$, we can compute $\mathbf{M}$ and $\mathbf{S}$, and obtain a linear system $\mathbf{Ax} = \mathbf{b}$.

Extension to Vector Field Processing

- We extend the formulation of gradient-domain processing to vector fields
- Using the duality between 1-forms and vector fields, we look for a 1-form ω on a triangle mesh $\mathcal{M}$ by minimizing

$$E(\omega) = \boxed{||\omega - \nu||^2} + \alpha \boxed{(||d\omega - \mu||^2 + ||\delta\omega - \psi||^2)}$$

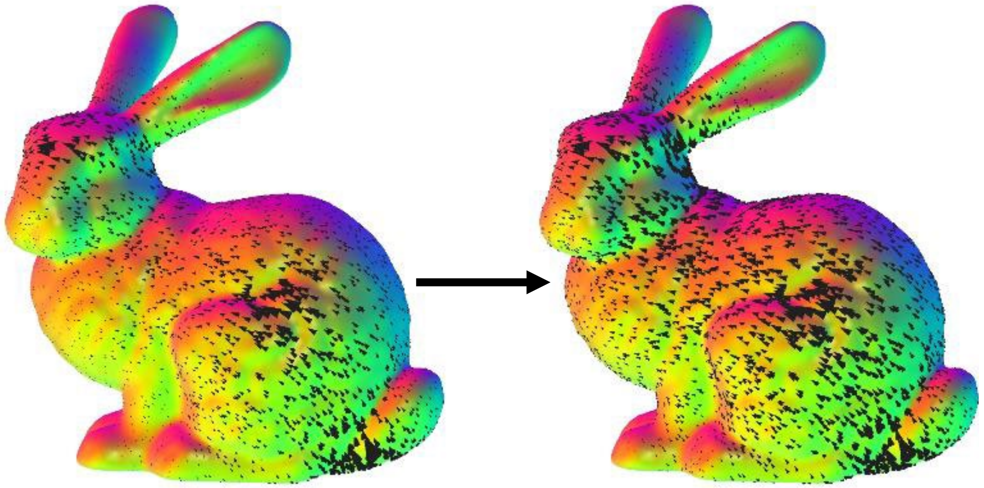
where

- ν: the target 1-form
- μ: the target 2-form
- ψ: the target 0-form
- α: the smoothness weight

Vector Field Gradient-Domain Processing

$$E(\omega) = \|\omega - \nu\|^2 + \alpha(\|d\omega - \mu\|^2 + \|\delta\omega - \psi\|^2)$$

Vector field diffusion



Logarithm map

Vector heat method

Vector Field Gradient-Domain Processing

- Discretizing using basis functions $\{\phi_e\}_{e\in E}$ gives:

$$\arg\min_{\omega} E(\omega) = \|\omega - \nu\|^2 + \alpha(\|d\omega - \mu\|^2 + \|\delta\omega - \psi\|^2)$$

$$\Downarrow$$

$$\arg\min_{\mathbf{w}} E(\mathbf{w}) = (\mathbf{w} - \mathbf{v})^\top \mathbf{M}(\mathbf{w} - \mathbf{v}) + \alpha(\mathbf{w}^\top \mathbf{S}\mathbf{w} - 2\mathbf{w}^\top \mathbf{u} + \mathbf{c})$$

$$\Downarrow$$

$$(\mathbf{M} + \alpha\mathbf{S})\mathbf{w} = \mathbf{M}\mathbf{v} + \alpha\mathbf{u}$$

Solving a
Linear System
$\mathbf{Ax} = \mathbf{b}$

where $\mathbf{w}, \mathbf{v}$ are coefficients vectors of ω, ν w.r.t. $\{\phi_e\}_{e\in E}$, $\mathbf{u}, \mathbf{c}$ are some constant in terms of μ, ψ

Vector Field Gradient-Domain Processing

$$(\mathbf{M} + \alpha\mathbf{S})\mathbf{w} = \mathbf{M}\mathbf{v} + \alpha\mathbf{u}$$

- $\mathbf{M}$: the 1-form mass matrix

$$\mathbf{M}_{e,e'} = \int_{\mathcal{M}} \langle \phi_e, \phi_{e'} \rangle \, dp$$

- $\boldsymbol{S}$: the 1-form stiffness matrix

$$\mathbf{S}_{e,e'} = \int_{\mathcal{M}} \langle d\phi_e, d\phi_{e'} \rangle \, dp + \int_{\mathcal{M}} \langle \delta\phi_e, \delta\phi_{e'} \rangle \, dp$$

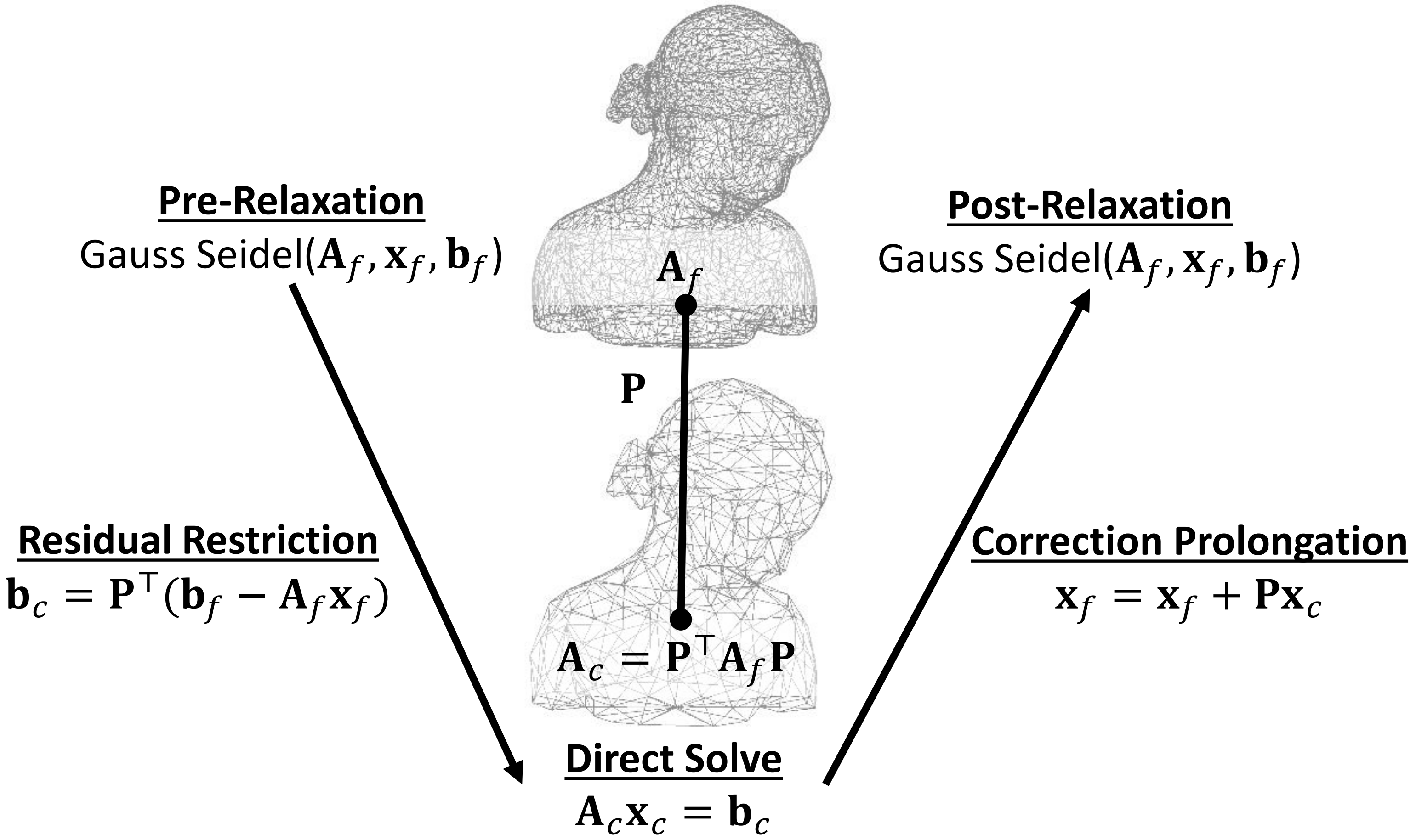
If we have 1-form basis functions $\{\phi_e\}_{e\in E}$ on $\mathcal{M}$, we can compute $\mathbf{M}$ and $\boldsymbol{S}$, and obtain a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Given a linear system $\mathbf{Ax} = \mathbf{b}$, how do we solve it efficiently?

Multigrid Method (V-Cycle)

Given:

- $\mathbf{A}_f = \mathbf{A} \in \mathbb{R}^{n_f \times n_f}$
- $\mathbf{b}_f = \mathbf{b} \in \mathbb{R}^{n_f}$
- $\mathbf{P} \in \mathbb{R}^{n_f \times n_c}$
  - a prolongation matrix
  - $n_c < n_f$
- $\mathbf{x}_f \in \mathbb{R}^{n_f}$
  - a current estimate



If we have $\mathbf{P}$, we can solve $\mathbf{Ax} = \mathbf{b}$ using multigrid efficiently.

$$\arg\min_{\phi} E(\phi) = \|\phi - \psi\|^2 + \alpha\|d\phi - \omega\|^2$$

$$\arg\min_{\omega} E(\omega) = \|\omega - \nu\|^2 + \alpha(\|d\omega - \mu\|^2 + \|\delta\omega - \psi\|^2)$$

Which basis do we use?

$(\mathbf{Ax} = \mathbf{b})$

How do we define the prolongation matrix $\mathbf{P}$?

(Solve $\mathbf{Ax} = \mathbf{b}$ using multigrid)

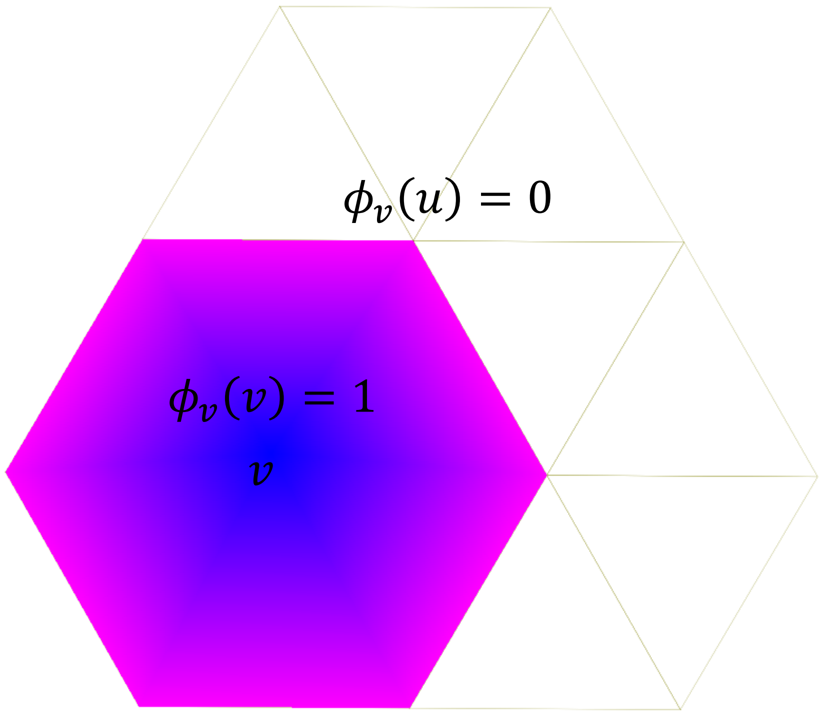
Scalar Field Basis Functions

- $\{\phi_v\}$: hat basis functions at each vertex
  - Piecewise linear
  - Lagrange interpolant property

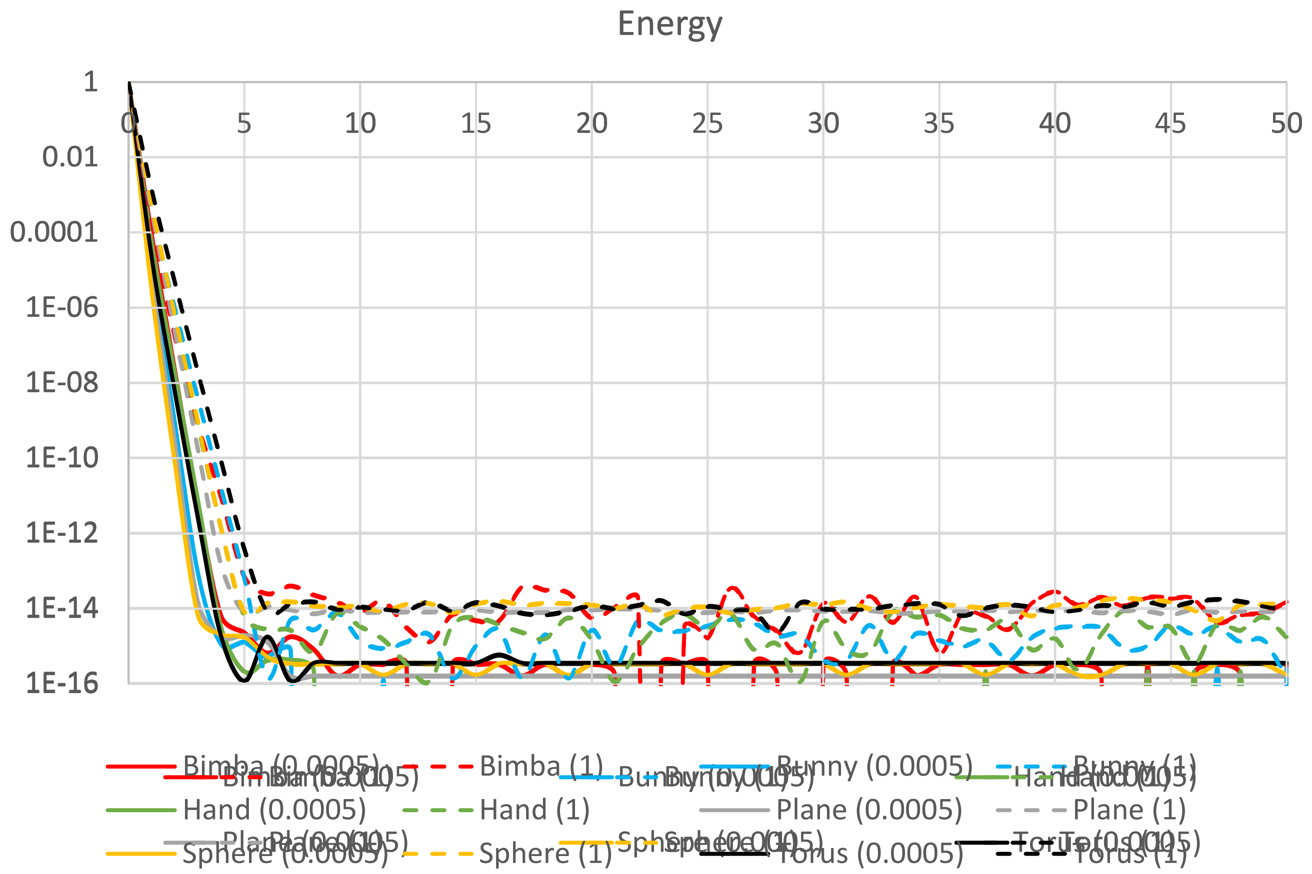
$$\phi_v(u) = \delta_{uv}$$

  - Partition of unity

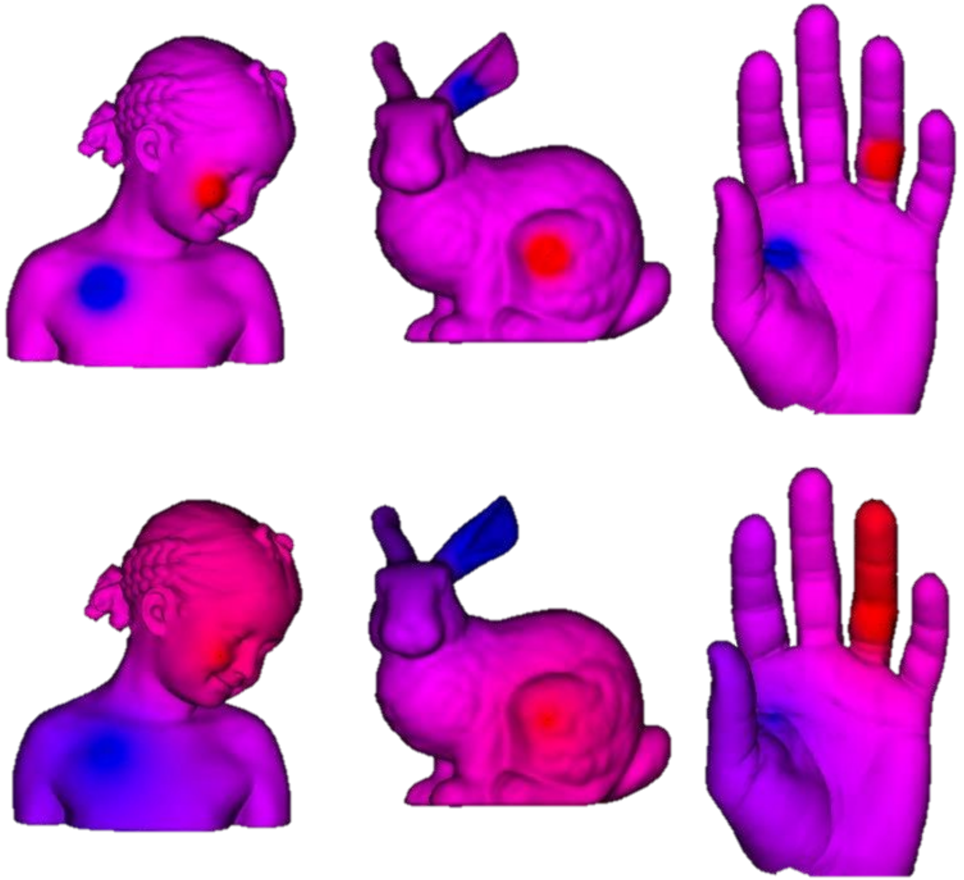
$$\sum_v \phi_v = 1$$



Scalar Field Diffusion Multigrid Convergence



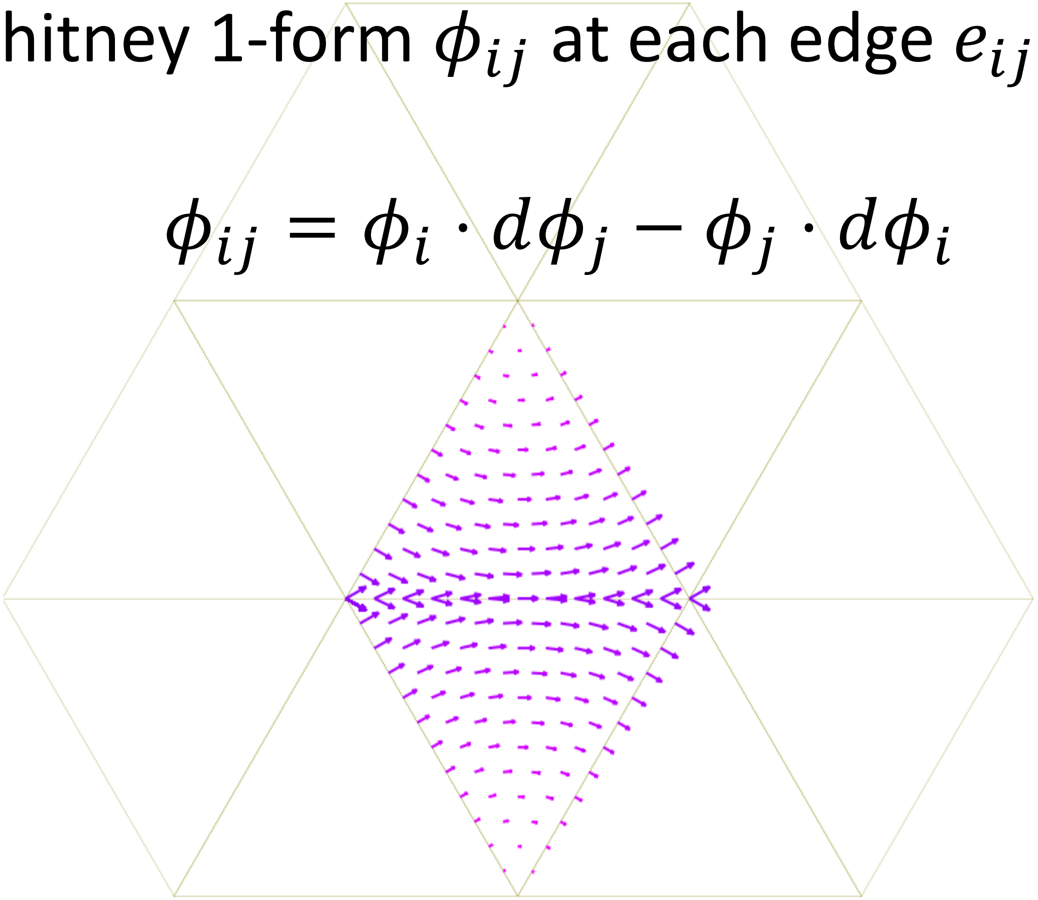
Minimizing $\|\phi - \psi\|^2 + \alpha\|d\phi\|^2$ for $\alpha = 0.0005$ (top) and $\alpha = 1$ (bottom)



Whitney Basis Functions

- On a triangle mesh, $\mathcal{M} = (V, E, T)$, given 0-form basis functions $\{\phi_i\}_{v_i \in V}$, the Whitney 1-form ϕ_{ij} at each edge $e_{ij} = (v_i, v_j) \in E$ is defined as

$$\phi_{ij} = \phi_i \cdot d\phi_j - \phi_j \cdot d\phi_i$$



1-Form Prolongation Matrix

Given:

- $\mathcal{M}_f = \left(V_f, E_f, T_f\right), \mathcal{M}_c = (V_c, E_c, T_c)$: the fine and coarse meshes
- 0-form prolongation matrix $\mathbf{P}^0 \in \mathbb{R}^{|V_f| \times |V_c|}$, which implies

$$\phi_i^c = \sum_m \mathbf{P}^0_{m,i} \phi_m^f$$

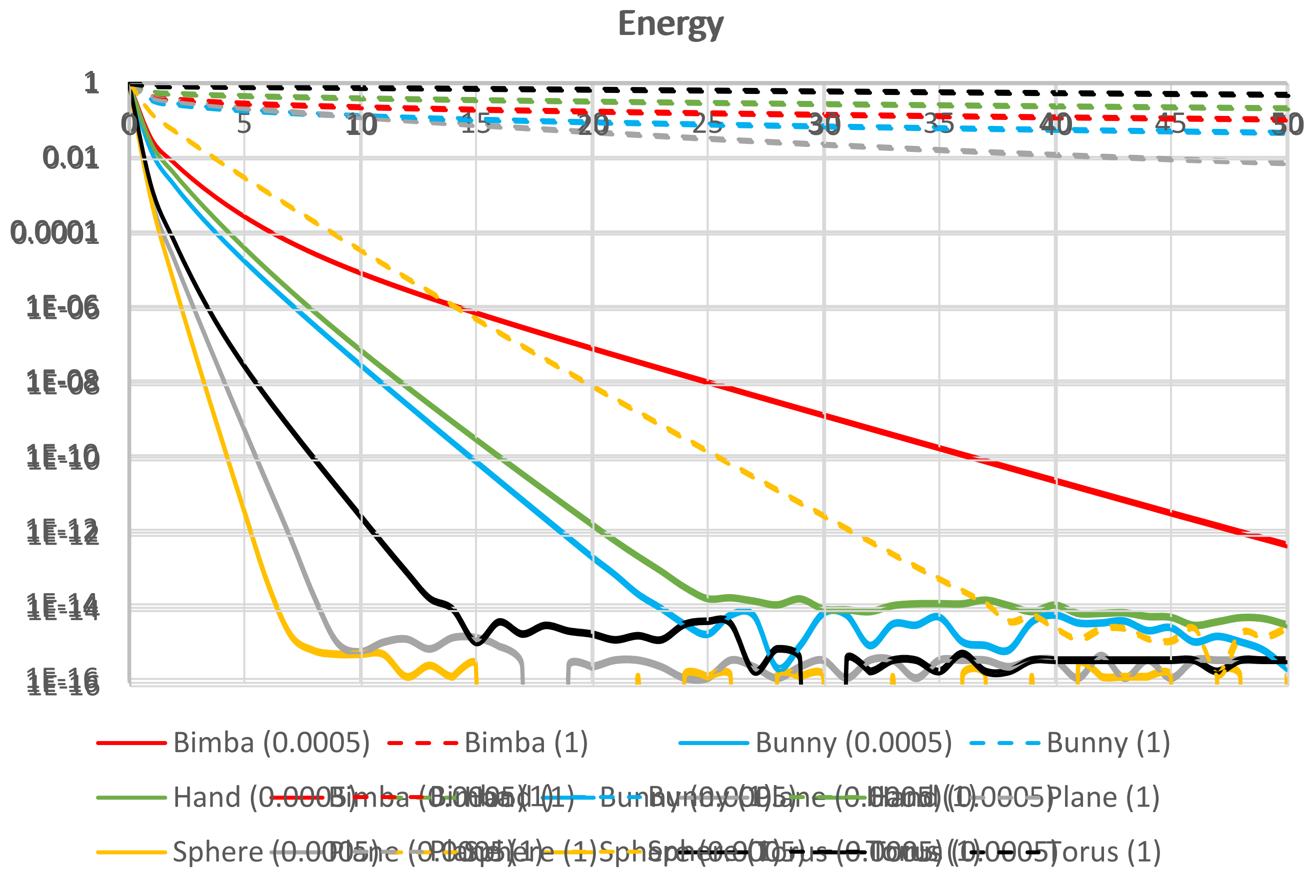
Goal:

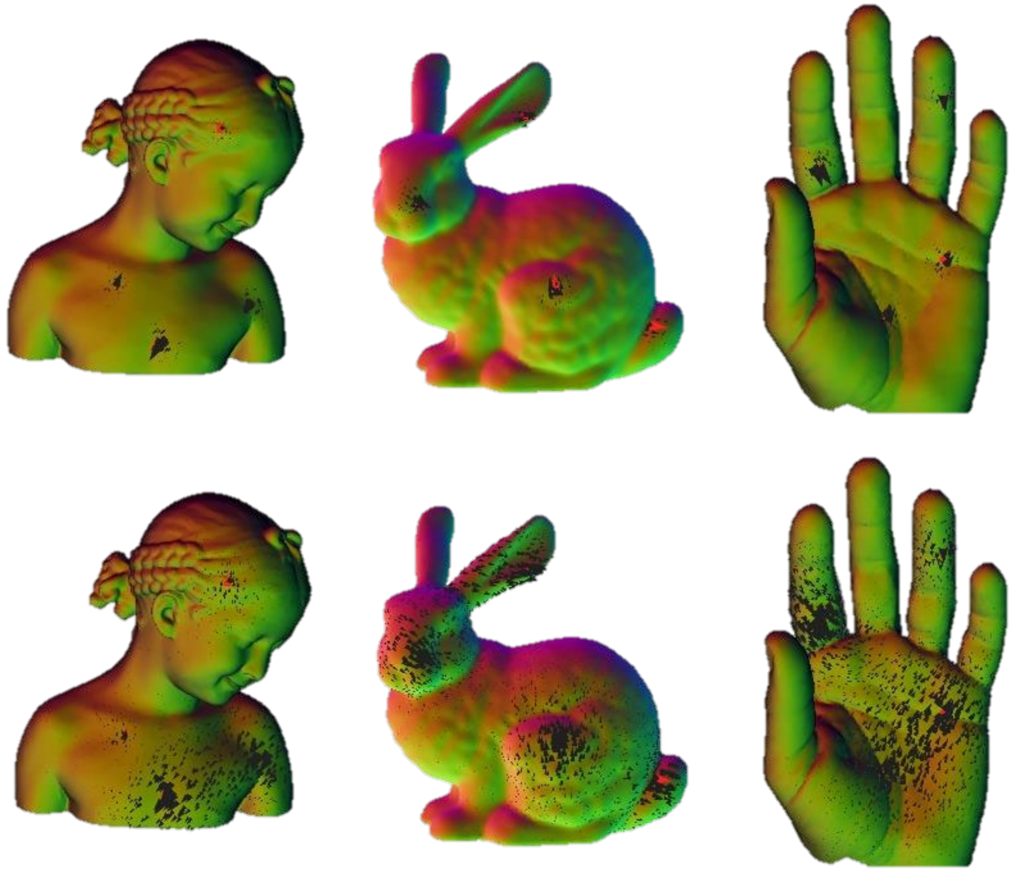
- Define 1-form prolongation, $\mathbf{P}^1 \in \mathbb{R}^{|E_f| \times |E_c|}$, using Whitney 1-form definition,

$$\phi_{ij}^c = \phi_i^c \cdot d\phi_j^c - \phi_j^c \cdot d\phi_i^c = \sum_{mn} \overbrace{\left(\mathbf{P}^0_{m,i}\mathbf{P}^0_{n,j} - \mathbf{P}^0_{n,i}\mathbf{P}^0_{m,j}\right)}^{\mathbf{P}^1_{mn,ij}} \phi_{mn}^f$$

Vector Field Diffusion Multigrid Convergence

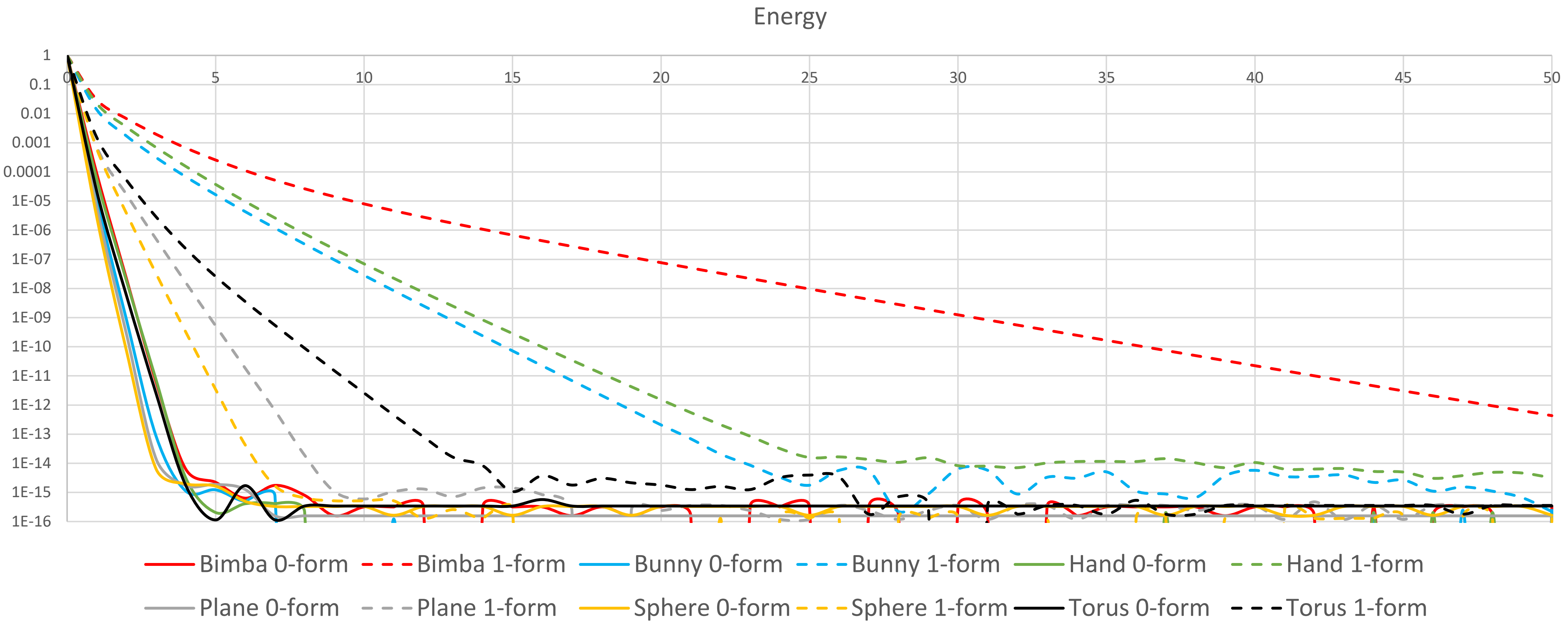
Minimizing$\|\omega - \nu\|^2 + \alpha(\|d\omega\|^2 + \|\delta\omega\|^2)$
for $\alpha = 0.0005$ (top) and $\alpha = 1$ (bottom)





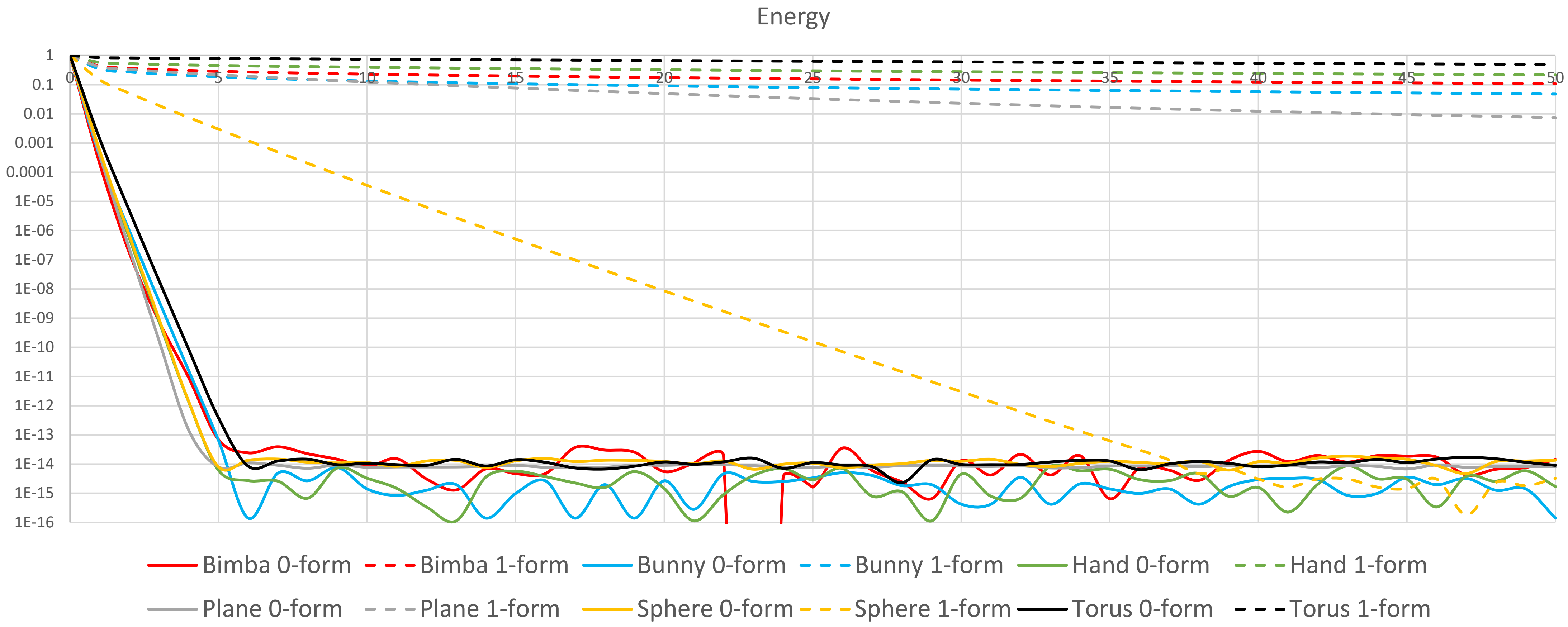
Comparison 0-Form and 1-Form Smoothing

Minimizing $\|\phi - \psi\|^2 + \alpha\|d\phi\|^2$ comparing to $\|\omega - \nu\|^2 + \alpha(\|d\omega\|^2 + \|\delta\omega\|^2)$ for $\alpha = 0.0005$



Comparison 0-Form and 1-Form Smoothing

Minimizing $\|\phi - \psi\|^2 + \alpha\|d\phi\|^2$ comparing to $\|\omega - \nu\|^2 + \alpha(\|d\omega\|^2 + \|\delta\omega\|^2)$ for $\alpha = 1$



Inspiration: Krylov Subspace Method

Given:

- $\mathbf{A}$: a system matrix
- $\mathbf{b}$: a right hand side

Goal:

- Construct the Krylov subspace $K_n \coloneqq \{\mathbf{b}, \mathbf{Ab}, \mathbf{A}^2\mathbf{b}, \cdots \mathbf{A}^{n-1}\mathbf{b}\}$
- Find $\mathbf{x} \in K_n$ that best solves $\mathbf{Ax} = \mathbf{b}$

Krylov Subspace Method (V-Cycle)

Given:

- $\mathbf{c}_0$: the current solution estimate
- $\mathbf{c}_i$: the computed correction at the ith step of a V-Cycle
- $\mathbf{P}_i$: the prolongation that prolongs $\mathbf{c}_i$ to the finest level

Goal:

- Let $\mathbf{P}_0$ be an identity, define a solution subspace $K = \{\mathbf{P}_i \mathbf{c}_i\}$
- Let $\mathbf{x}^*$ is the ground-truth such that $\mathbf{A}\mathbf{x}^* = \mathbf{b}$, solve for the best solution in K

$$\arg\min_{\mathbf{x} \in K} E(\mathbf{x}) = (\mathbf{x} - \mathbf{x}^*)^\top \mathbf{A} (\mathbf{x} - \mathbf{x}^*)$$

- Let $\boldsymbol{\alpha}$ be the coefficients, i.e. $\mathbf{x} = \sum_i \alpha_i \mathbf{P}_i \mathbf{c}_i$, minimizing $E(\mathbf{x})$ is to solve

$$\left[(\mathbf{P}_i \mathbf{c}_i)^\top \mathbf{A} \mathbf{P}_j \mathbf{c}_j\right] \boldsymbol{\alpha} = \left[(\mathbf{P}_i \mathbf{c}_i)^\top \mathbf{b}\right]$$

Krylov Subspace Method (Iterations)

Given:

- $\mathbf{x}_i$: the estimated solution at the ith iteration

Goal:

- Define a solution subspace $K = \{\mathbf{x}_i\}$
- Let $\mathbf{x}^*$ is the ground-truth such that $\mathbf{A}\mathbf{x}^* = \mathbf{b}$, solve in K

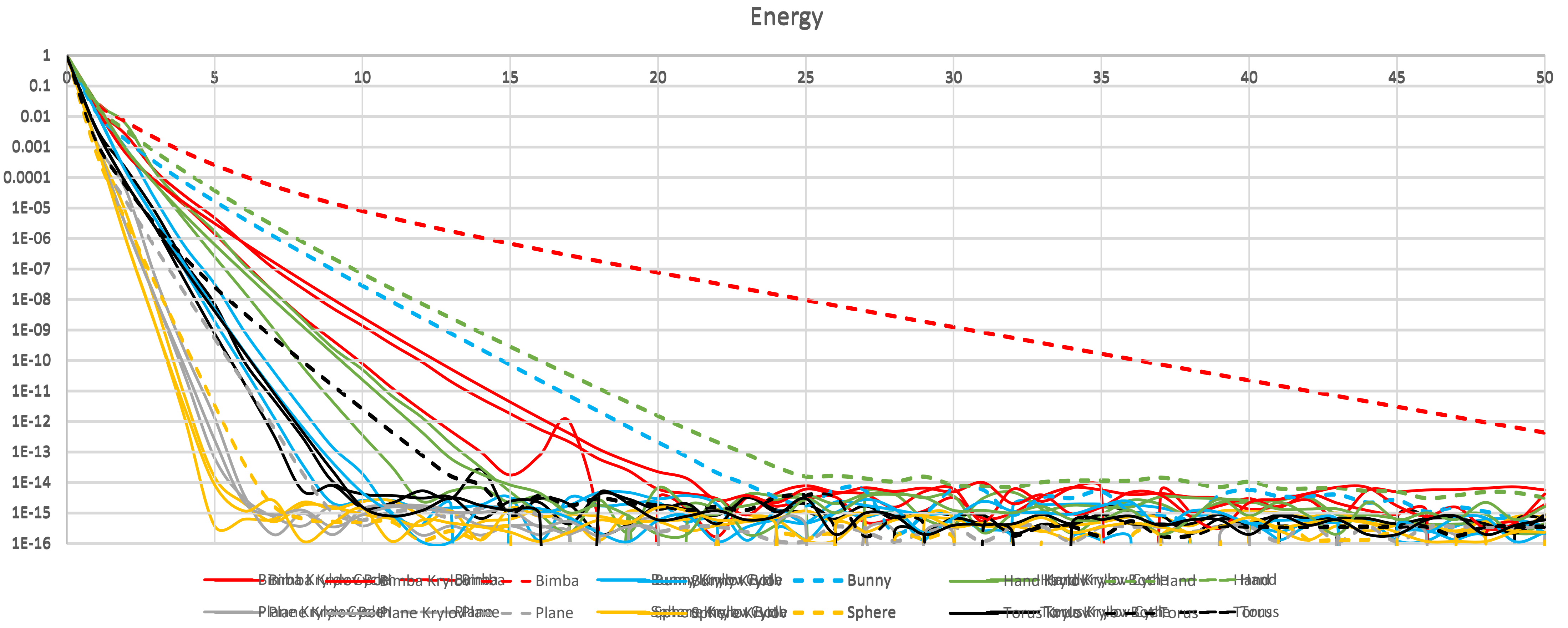
$$\arg\min_{\mathbf{x}\in K} E(\mathbf{x}) = (\mathbf{x} - \mathbf{x}^*)^\top \mathbf{A}(\mathbf{x} - \mathbf{x}^*)$$

- Let $\boldsymbol{\alpha}$ be the coefficients, i.e. $\mathbf{x} = \sum_i \alpha_i \mathbf{x}_i$, minimizing $E(\mathbf{x})$ is to solve

$$\left[\mathbf{x}_i{}^\top \mathbf{A}\mathbf{x}_j\right]\boldsymbol{\alpha} = \left[\mathbf{x}_i{}^\top \mathbf{b}\right]$$

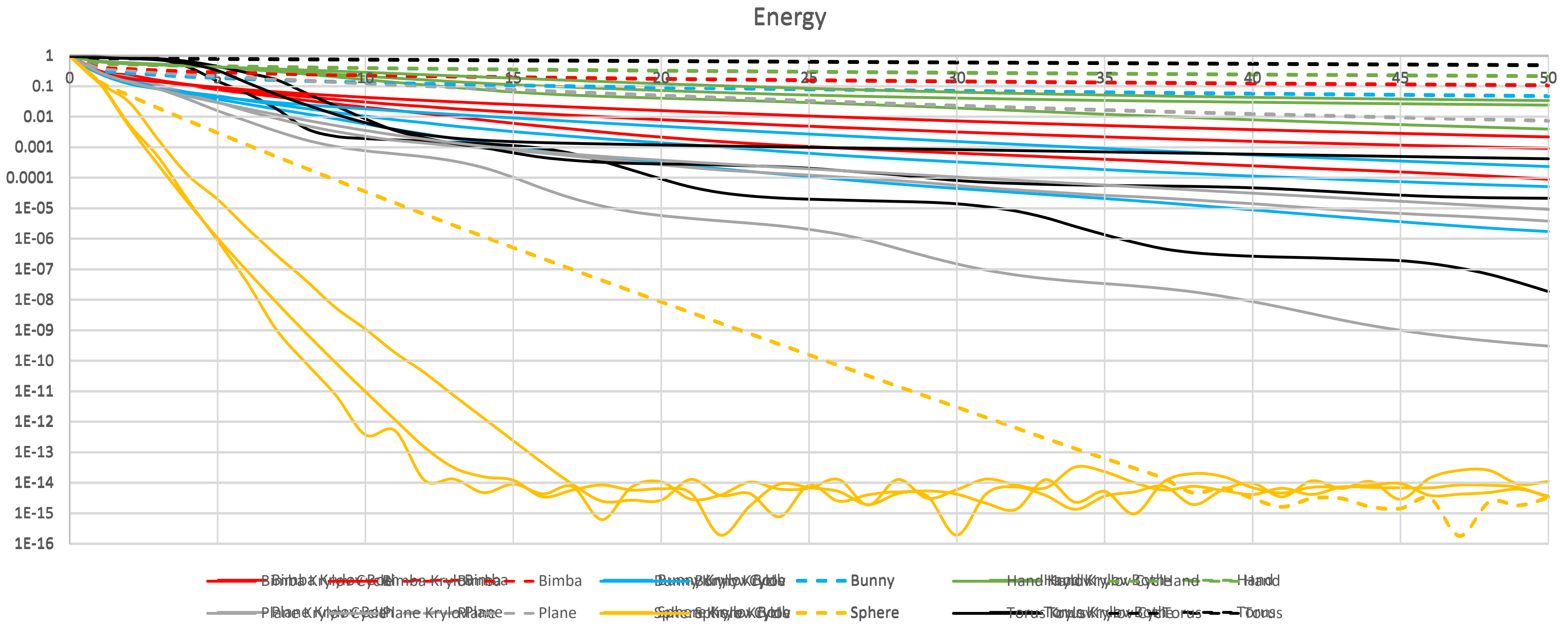
Convergence using Krylov Subspace Update

Minimizing $\|\omega - \nu\|^2 + \alpha(\|d\omega\|^2 + \|\delta\omega\|^2)$ for $\alpha = 0.0005$

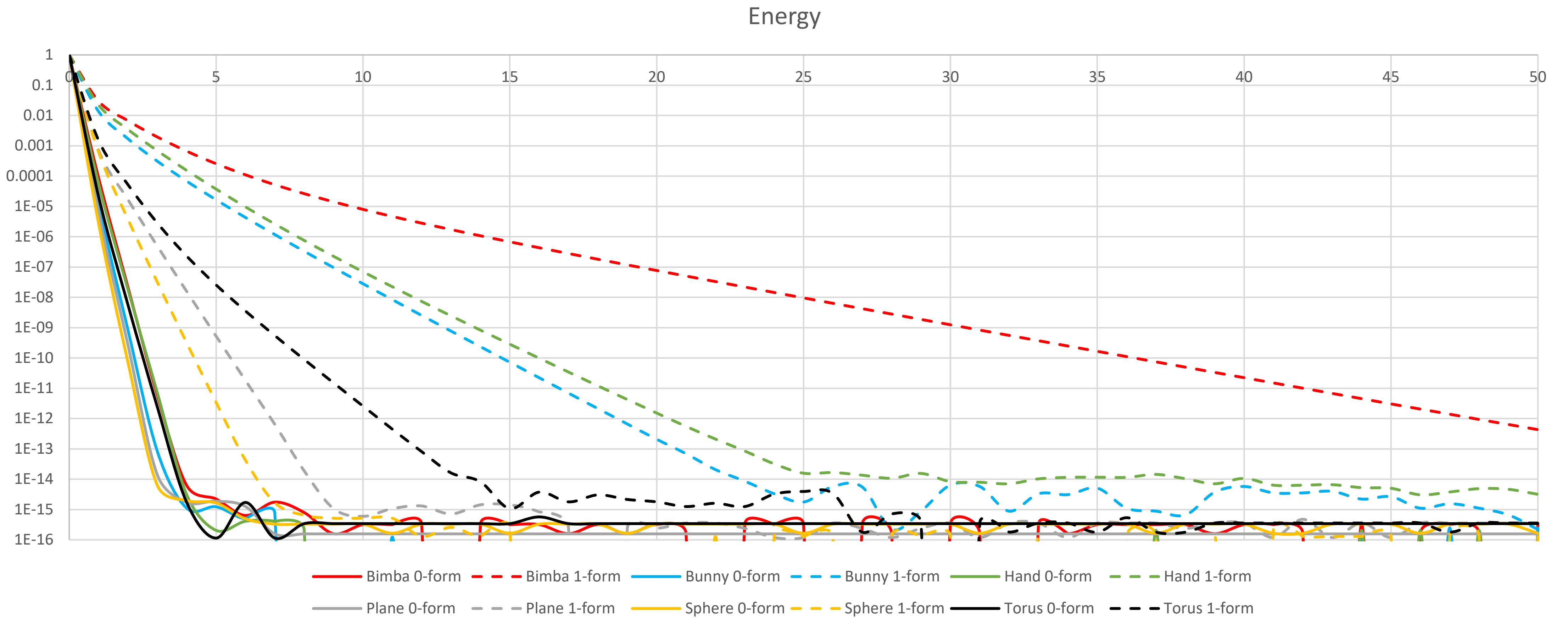


Convergence using Krylov Subspace Update

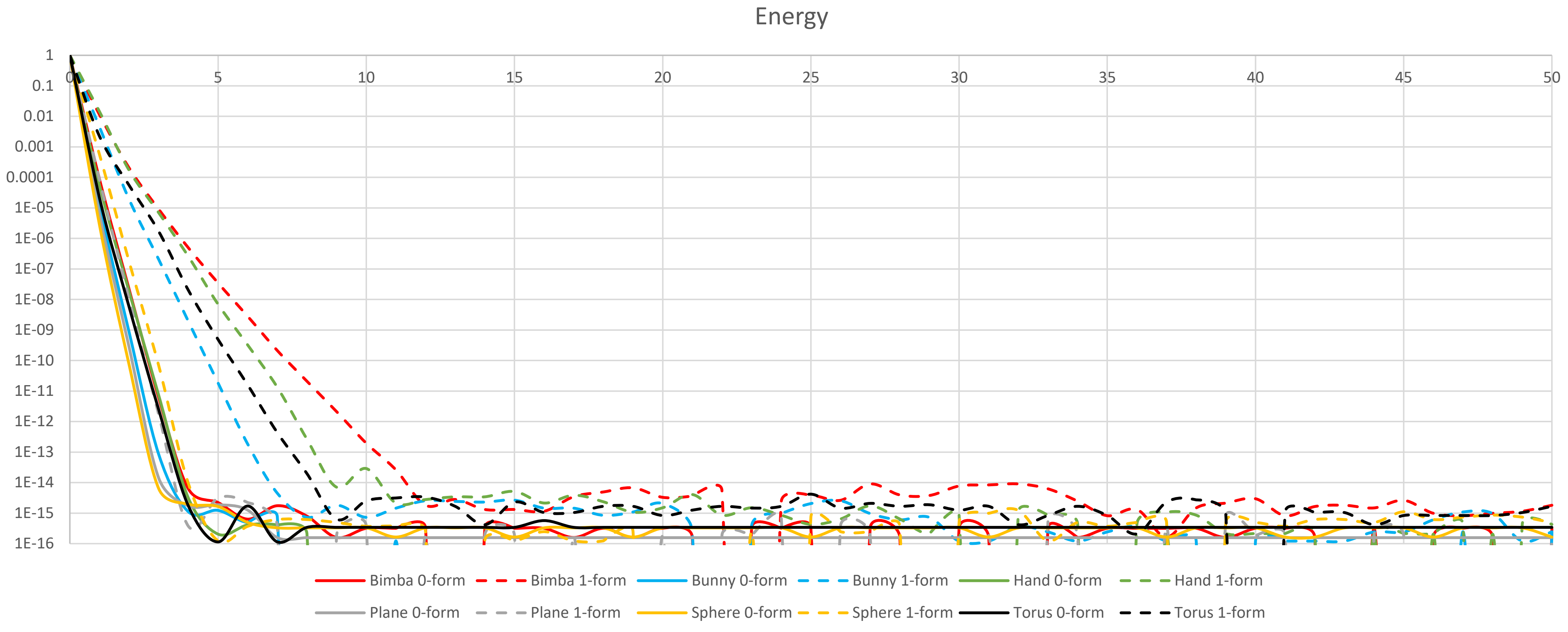
Minimizing $\|\omega - \nu\|^2 + \alpha(\|d\omega\|^2 + \|\delta\omega\|^2)$ for $\alpha = 1$



A Typical V-Cycle: 0-Form vs 1-Form



Current Results: 0-Form vs 1-Form



Conclusions

1-form (k-form) gradient-domain processing energy:

$$E(\omega) = \|\omega - \nu\|^2 + \alpha(\|d\omega - \mu\|^2 + \|\delta\omega - \psi\|^2)$$

Given a basis, discretize using mass and stiffness matrices:

$$E(\mathbf{w}) \; = (\mathbf{w} - \mathbf{v})^\top \mathbf{M}(\mathbf{w} - \mathbf{v}) + \alpha(\mathbf{w}^\top \mathbf{S}\mathbf{w} - 2\mathbf{w}^\top \mathbf{u} + \mathbf{c})$$

$$\Downarrow$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

1-form prolongation matrix derived from Whitney 1-form definition:

$$\phi_{ij}^c = \phi_i^c \cdot d\phi_j^c - \phi_j^c \cdot d\phi_i^c = \sum_{mn} \overbrace{\left(\mathbf{P}_{m,i}^0 \mathbf{P}_{n,j}^0 - \mathbf{P}_{n,i}^0 \mathbf{P}_{m,j}^0\right)}^{\mathbf{P}_{mn,ij}^1} \phi_{mn}^f$$

Conclusions

Challenges:

- Slower convergence
- The system matrix is poorly conditioned for big α
- The coarse system constructed using ***1-form prolongation matrix*** does not mainly solve for the low frequency part of the fine system

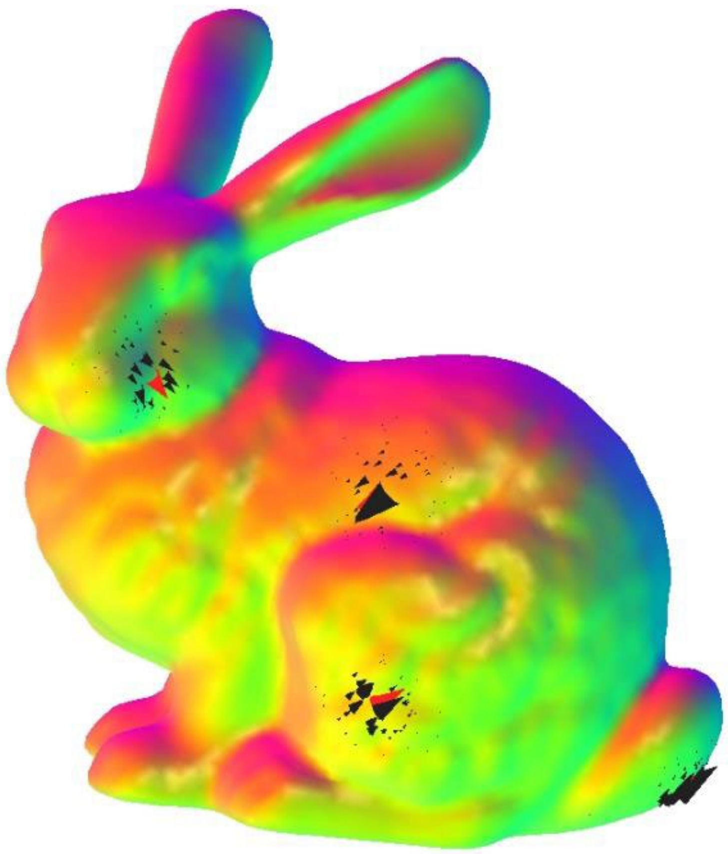
Solutions:

- ***Krylov subspace update***

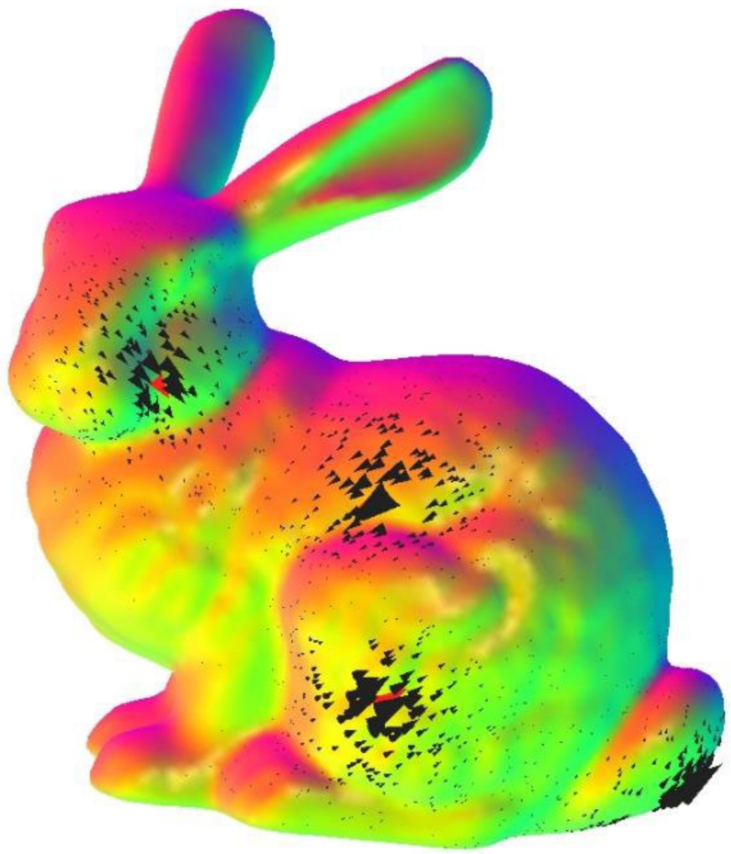
Thank you!

Applications – Vector Field Interpolation

Iteration 1

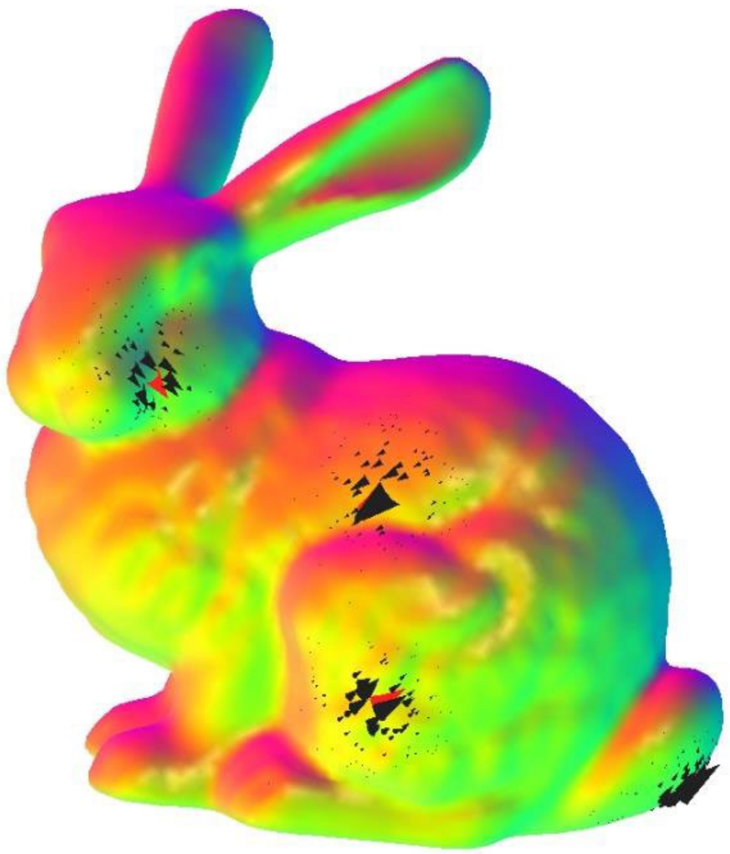


Direct Solver

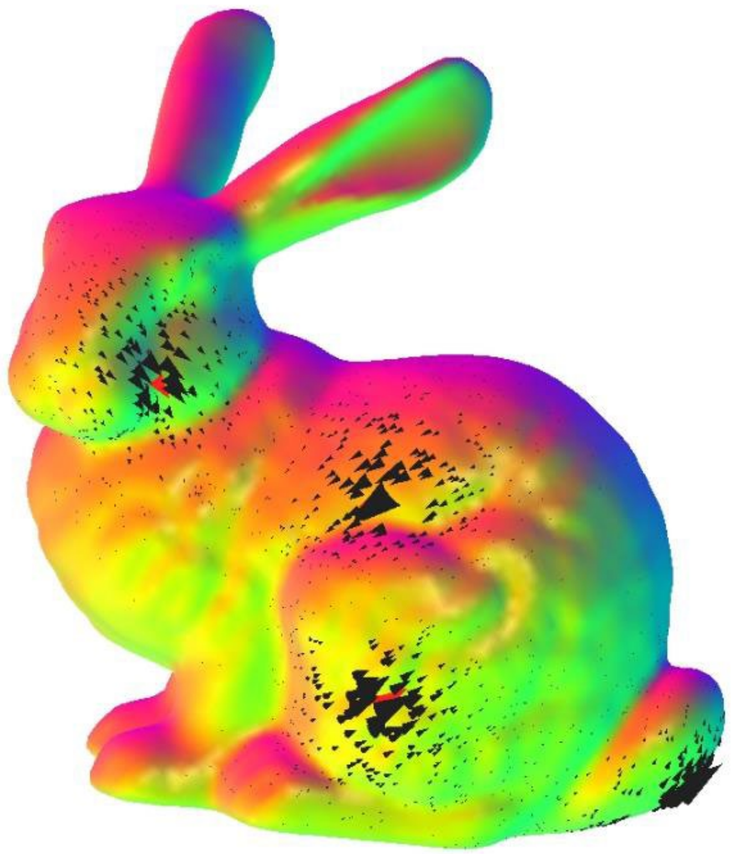


Applications – Vector Field Interpolation

Iteration 2

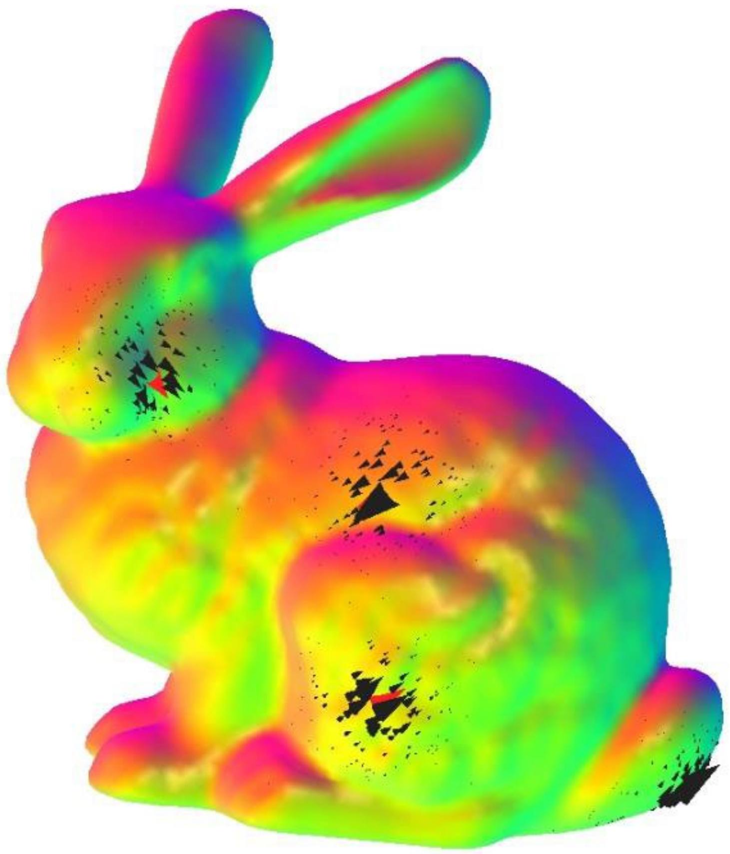


Direct Solver

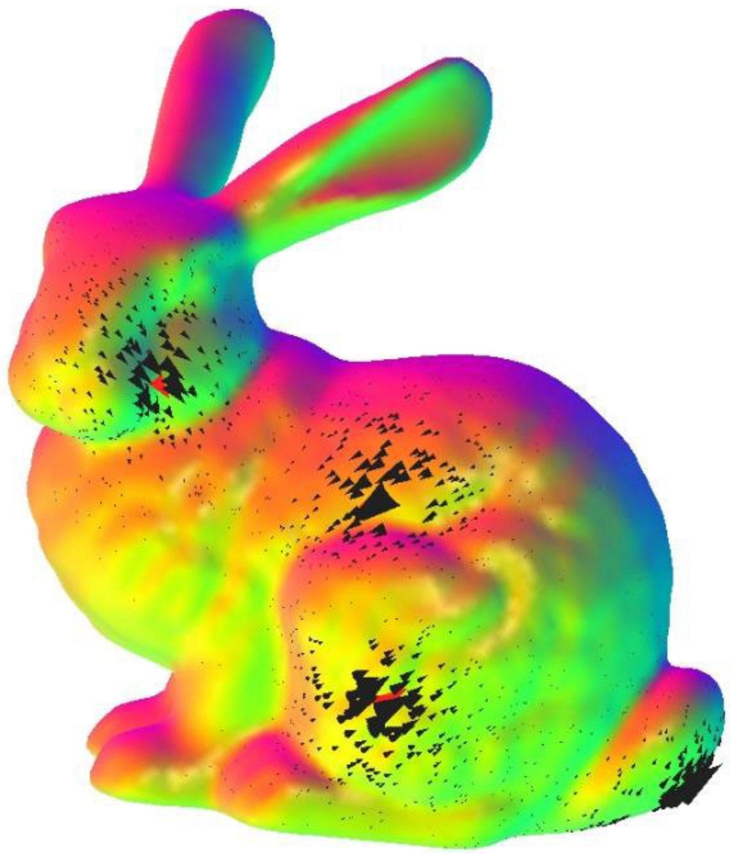


Applications – Vector Field Interpolation

Iteration 3

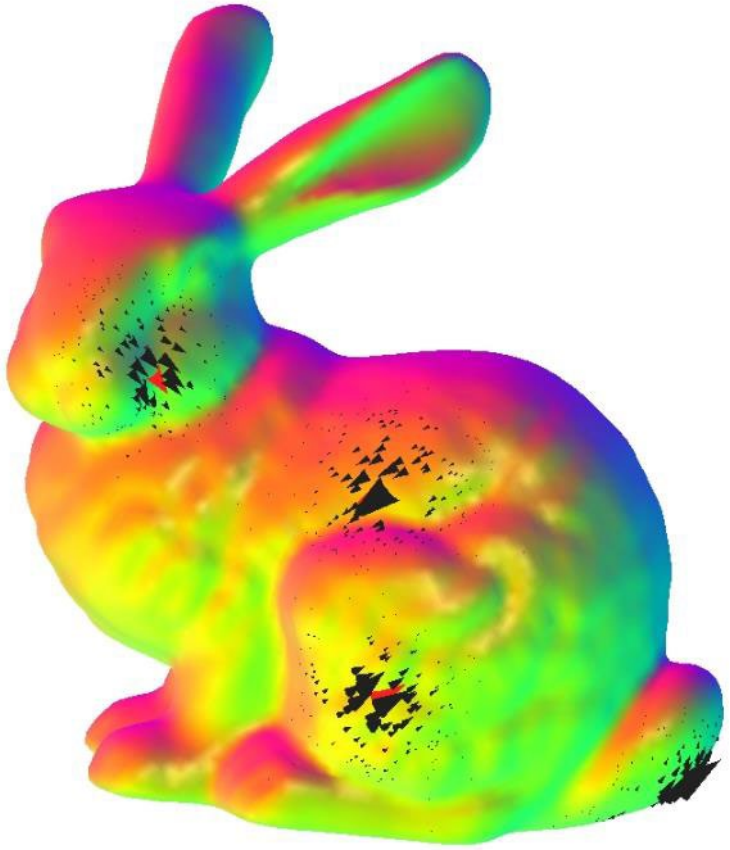


Direct Solver

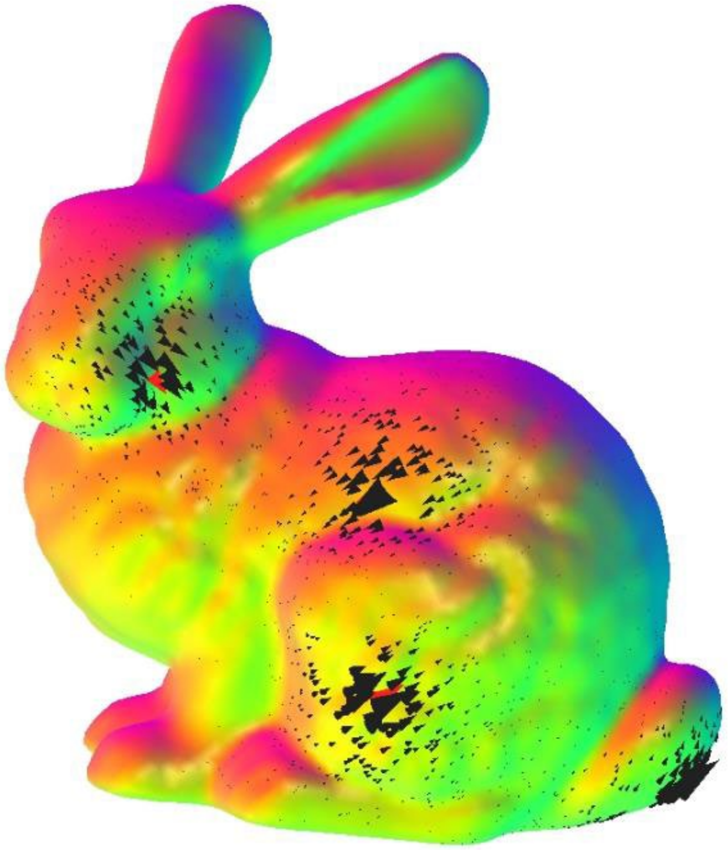


Applications – Vector Field Interpolation

Iteration 4

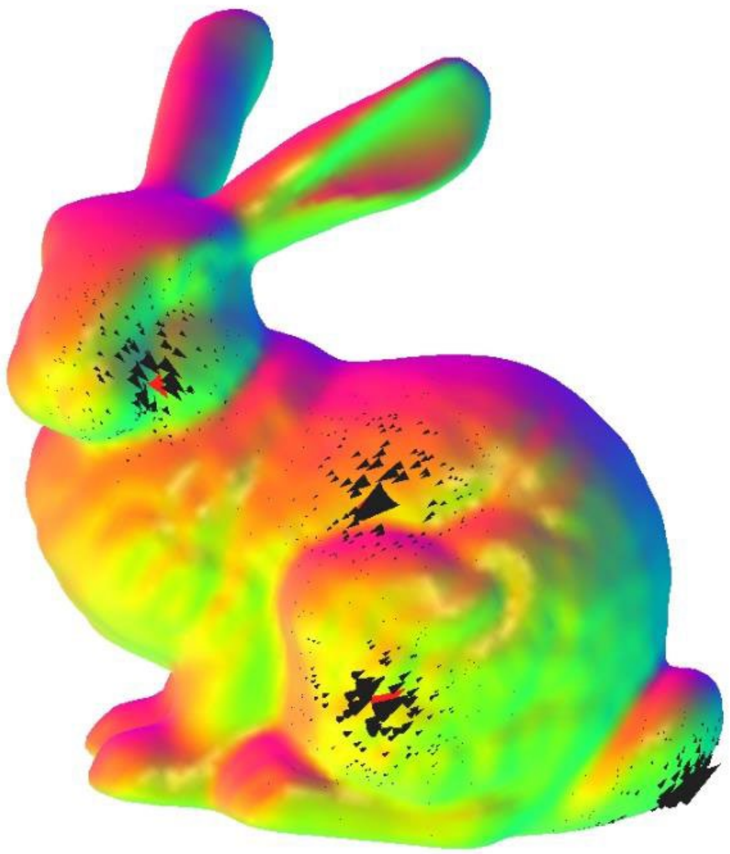


Direct Solver

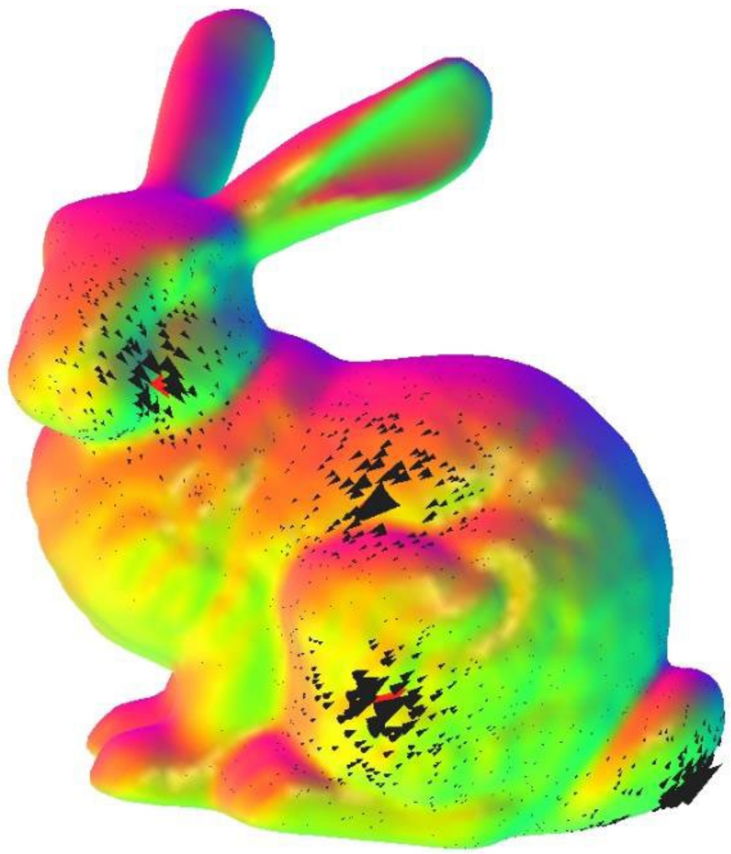


Applications – Vector Field Interpolation

Iteration 5

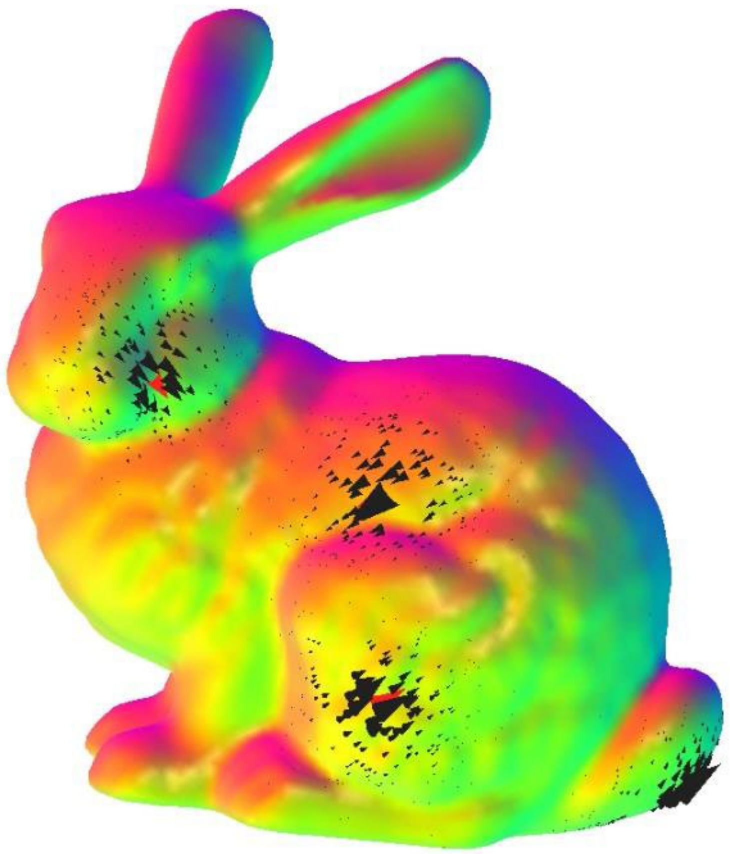


Direct Solver

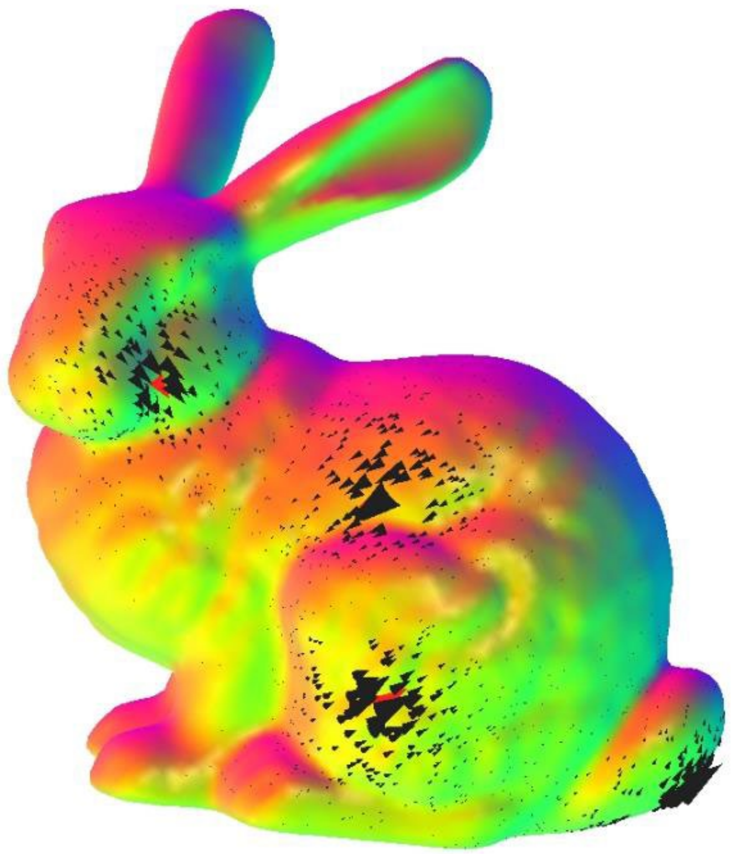


Applications – Vector Field Interpolation

Iteration 6

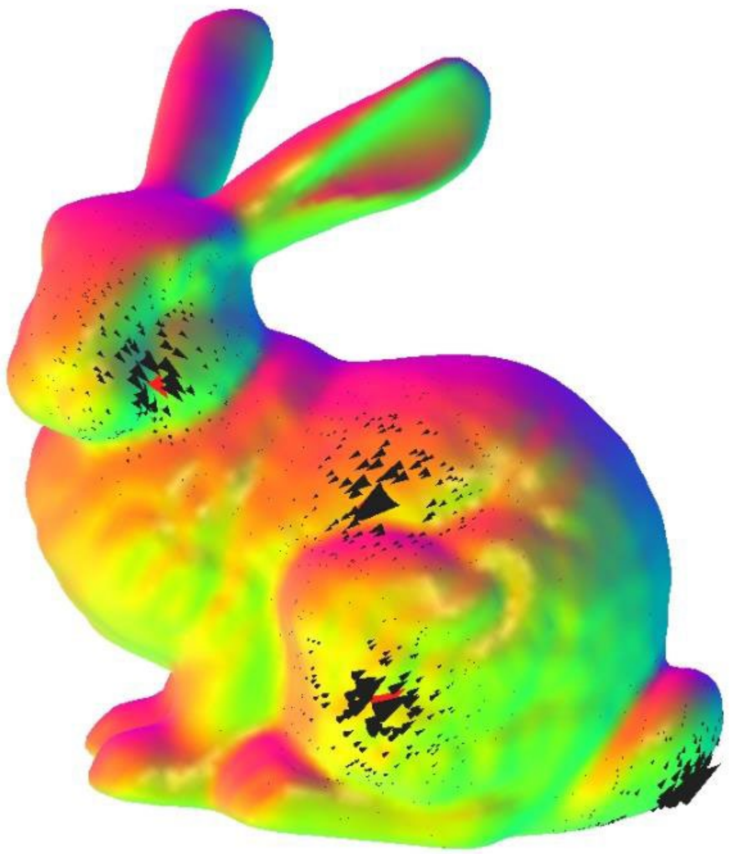


Direct Solver

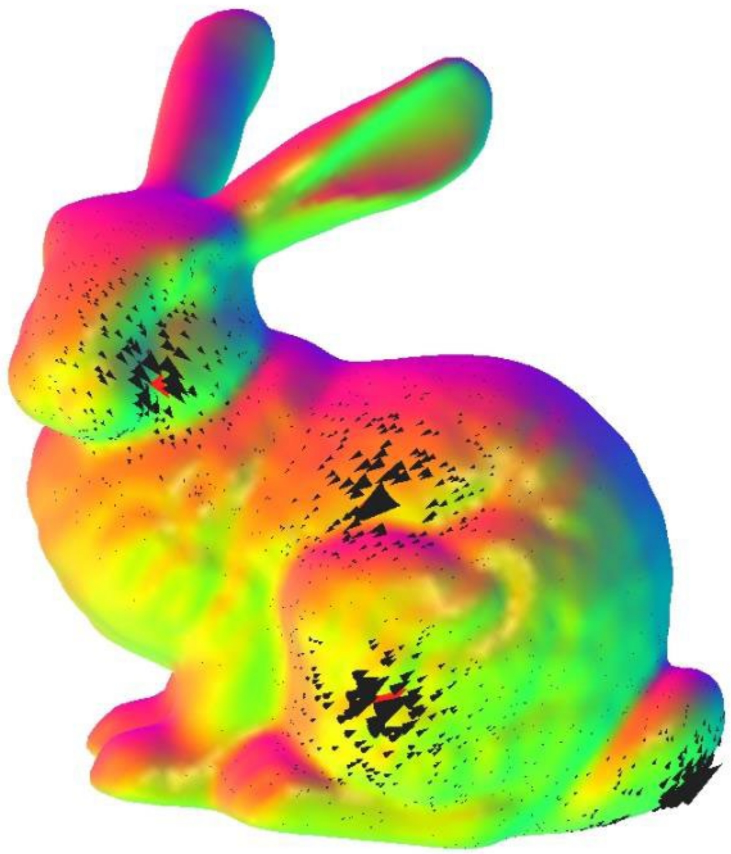


Applications – Vector Field Interpolation

Iteration 7

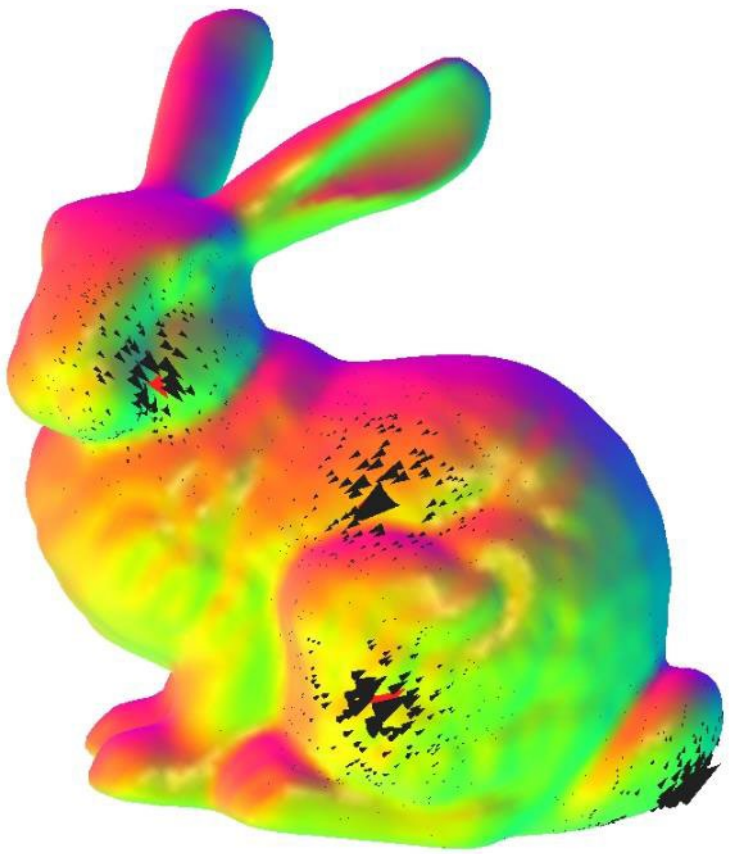


Direct Solver

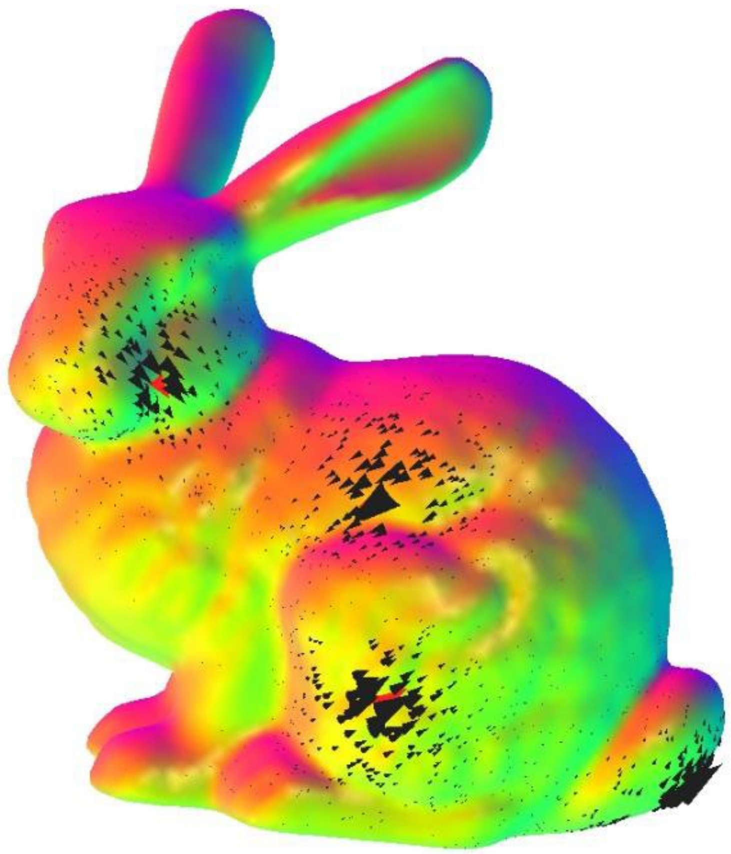


Applications – Vector Field Interpolation

Iteration 8

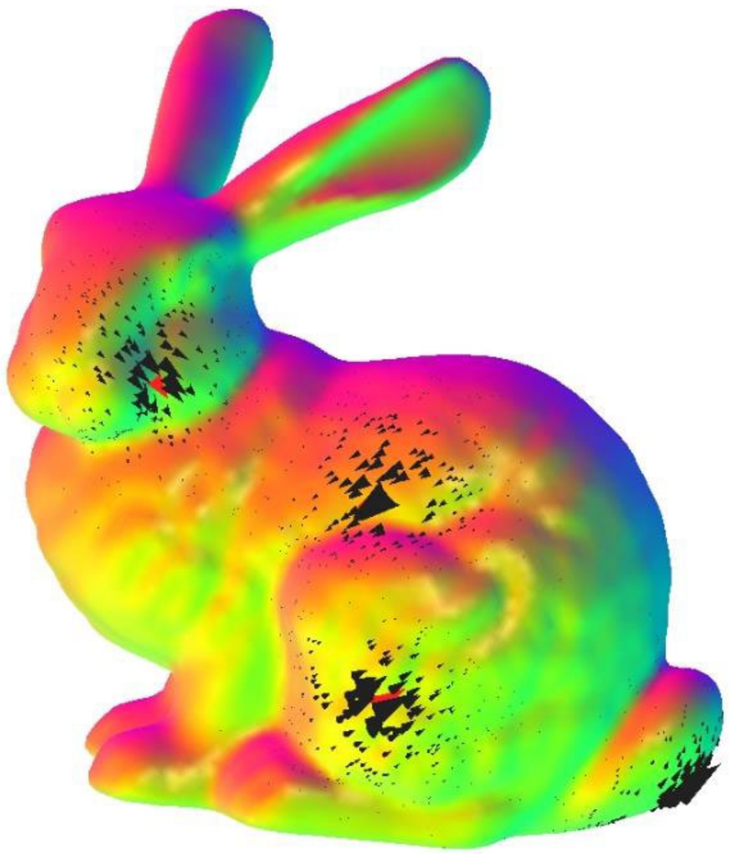


Direct Solver

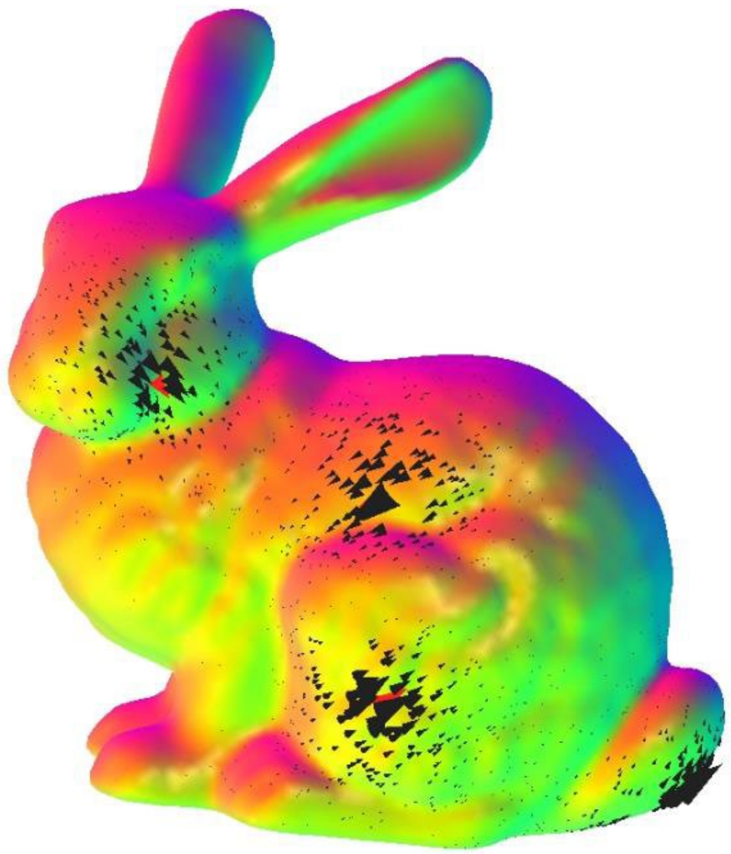


Applications – Vector Field Interpolation

Iteration 9

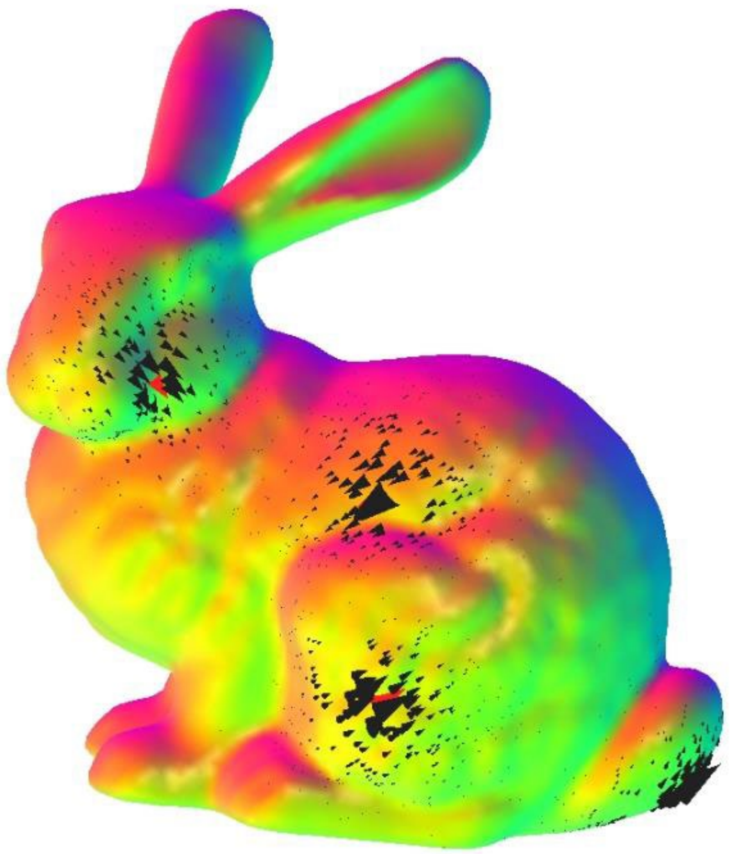


Direct Solver

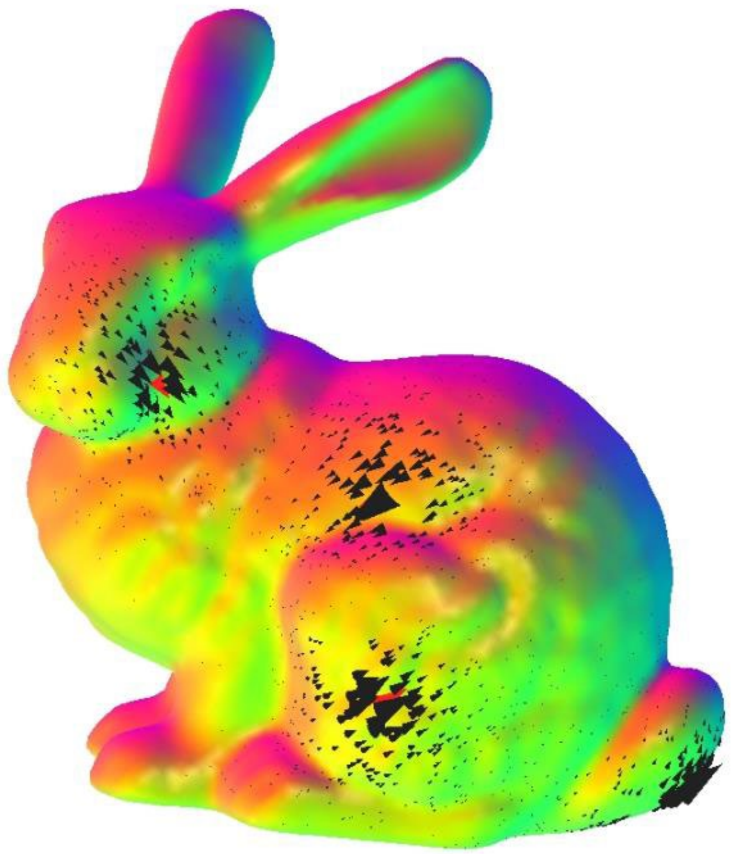


Applications – Vector Field Interpolation

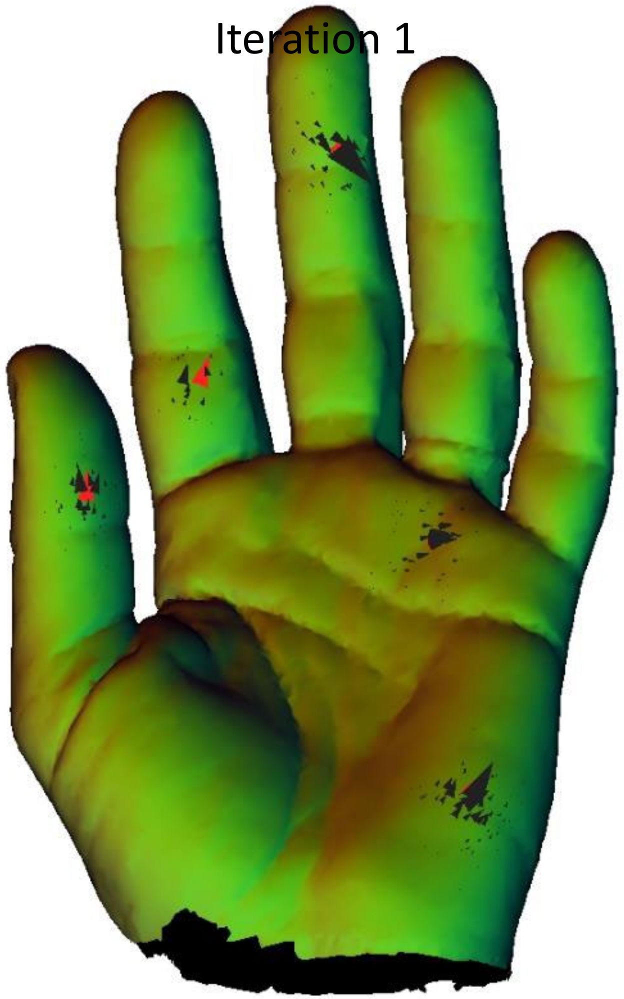
Iteration 10



Direct Solver

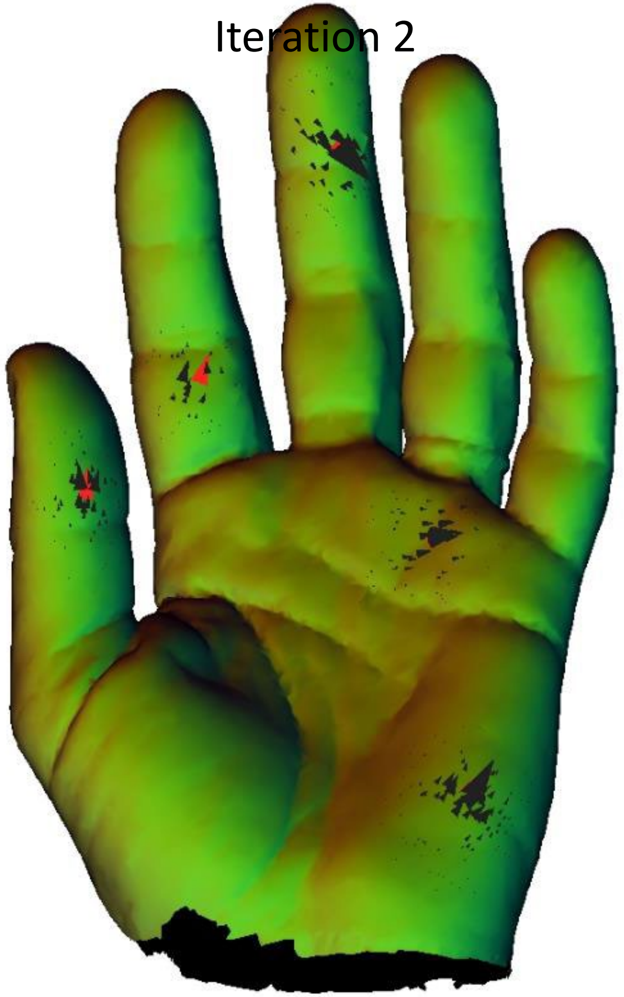


Applications – Vector Field Interpolation



Applications – Vector Field Interpolation

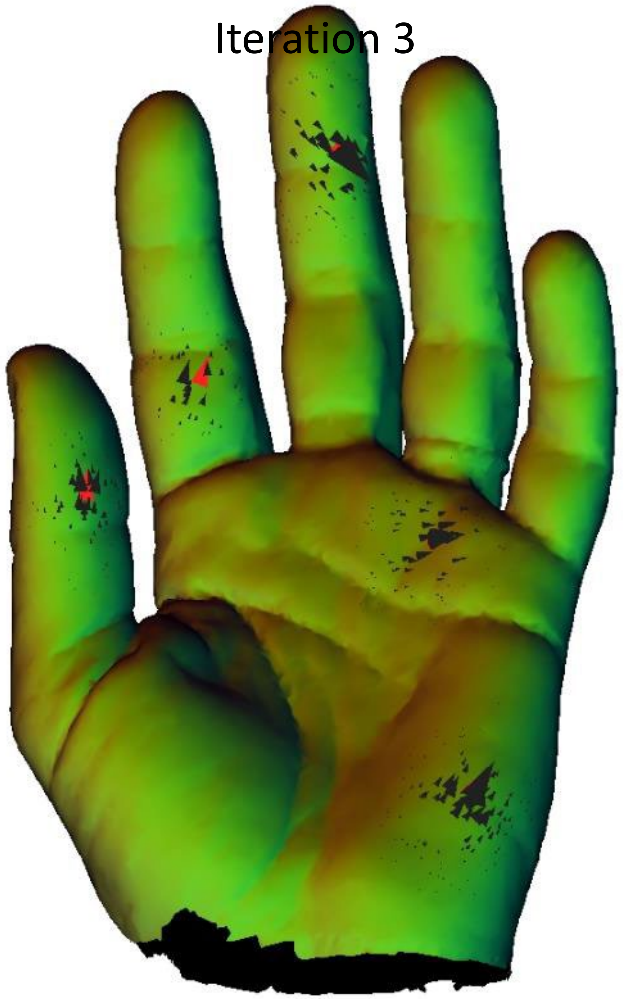
Iteration 2



Direct Solver

Applications – Vector Field Interpolation

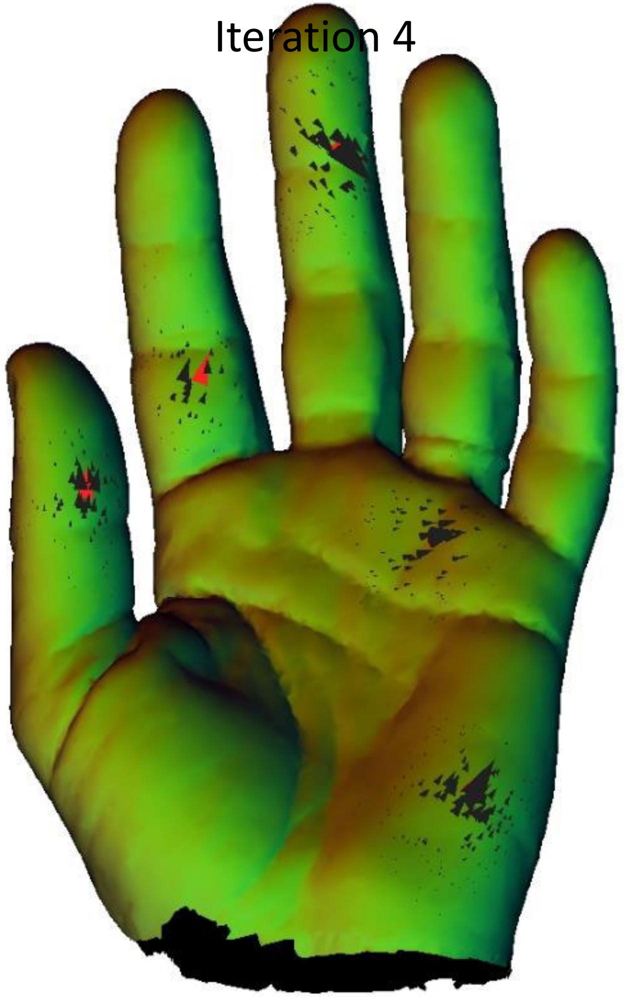
Iteration 3



Direct Solver

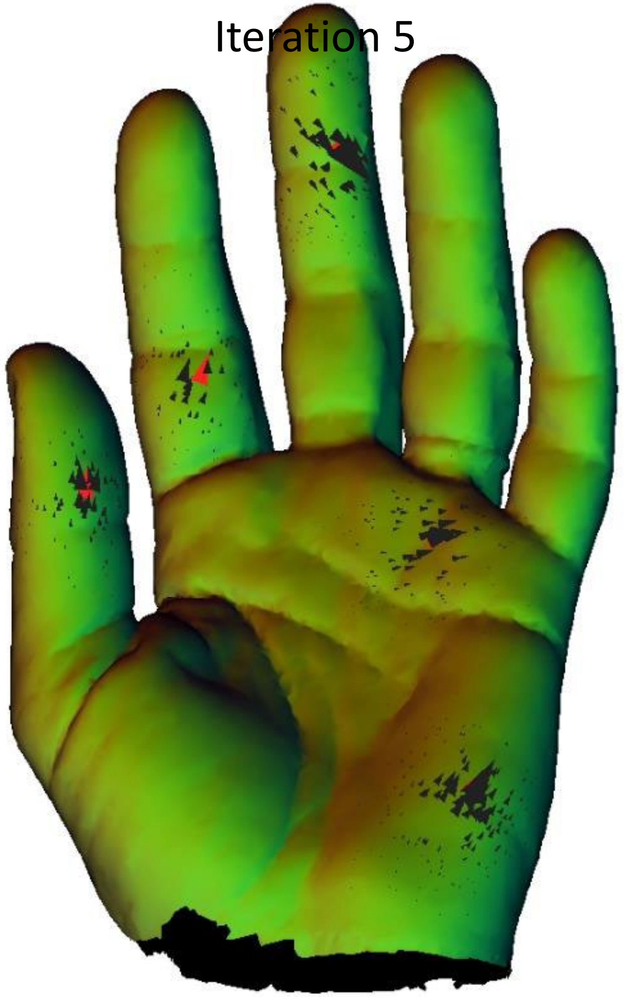
Applications – Vector Field Interpolation

Iteration 4



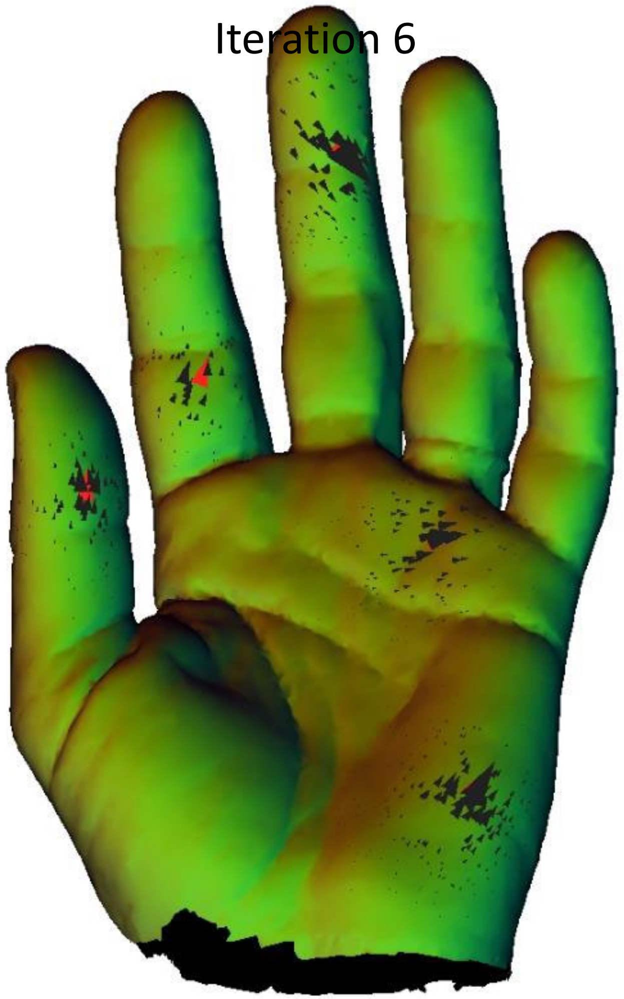
Direct Solver

Applications – Vector Field Interpolation



Applications – Vector Field Interpolation

Iteration 6



Direct Solver

Applications – Vector Field Interpolation

Applications – Vector Field Interpolation

Iteration 8

Direct Solver

Applications – Vector Field Interpolation

Iteration 9

Direct Solver

Applications – Vector Field Interpolation

Iteration 10

Direct Solver